A Review of Generalized and Unsharp Measurements

Humairah Bassa[1] **and Thomas Konrad**[1,2]

[1] School of Chemistry and Physics, University of KwaZulu-Natal, Durban 4000, South Africa
[2] National Institute of Theoretical Physics, Durban, South Africa

E-mail: `konradt@ukzn.ac.za`

Abstract. Unsharp measurements are a special class of generalized measurements whose experimental potential has not as yet been fully explored. Unlike the usual projective measurements which collapse the state of a quantum system into an eigenstate of the measured observable, unsharp measurements have a weaker influence on the state of the system. These types of measurements are advantageous when we need to gain information about a system but the disturbance must not exceed a certain level. It is therefore useful to study the set of unsharp measurements from first principles by reformulating the measurement postulate in terms of generalized measurements. The necessary requirements of an unsharp measurement are presented. A recent implementation scheme for unsharp measurements on trapped ions is reviewed.

1. Introduction

In 1932, John von Neumann [1] formulated a rigorous description of what happens to a quantum object when it is measured. The measurements he described are aptly named projective measurements since the measurement process projects the initial state of the system into an eigenstate of the measured observable. This eigenstate corresponds to the eigenvalue which was identical to the measurement outcome. For example, if we measured the energy of an atom in a projective measurement the outcome would be one of the quantized energies which are characteristic for that atom. As we can see these measurements are highly accurate since the measurement outcome is a definite and precise value which upon immediate repetition of the measurement would occur again. They are therefore also known as ideal or *sharp* measurements. However, the more accurate a measurement is, the greater is its disturbing influence on the measured system. We may sometimes require information about a quantum system but only a certain degree of disturbance may be acceptable. A possible example would be the real time estimation and monitoring of the Rabi oscillations of a single atom in a resonant driving field. Carrying out a sequence of projective measurements of the atomic energy would greatly disturb the system and in the continuum limit they would lead to the Quantum Zeno effect, i.e. they would freeze the dynamics of the system. An alternative are *unsharp* measurements which have a weaker influence on the state of the system but still convey some information about an associated ordinary observable, such as energy, position or spin. It has been shown that continuous unsharp measurement allows us to monitor the Rabi oscillations in the aforementioned case [2]. Remarkably, this is even possible in the presence of noise [3].

This article serves as a concise introduction to generalized and unsharp measurements and it is structured as follows. In Section 2, we reformulate the measurement postulate in terms generalized measurements which is the set of all measurements possible within the limit of quantum mechanics. The special properties of unsharp measurements are reviewed in Section 3 while Section 4 presents an implementation scheme for unsharp measurements on a trapped ion. A short discussion is given in Section 5.

2. Generalizing the measurement postulate

The measurement postulate, as described by von Neumann, fully characterizes a projective or sharp measurement and can be phrased as follows:

The result of a projective measurement of an observable A on a system prepared in the state ρ_{in} will be one of the eigenvalues of the observable defined by, $A|a\rangle = a|a\rangle$. The probability of obtaining measurement result "a" reads

$$p(a, \rho_{\text{in}}) = \text{Tr}[P_a \rho_{\text{in}}] \quad \text{where} \quad P_a = |a\rangle\langle a|, \tag{1}$$

and the change of the state due to the measurement is given by

$$\rho_{\text{in}} \to \rho_{\text{out}} = \frac{1}{p(a, \rho_{\text{in}})} P_a \rho_{\text{in}} P_a^{\dagger}. \tag{2}$$

The eigenstates of the observable form a complete orthonormal set thus we have the completeness relation $\sum_a P_a = \mathbb{1}$.

Projective measurements, however, only represent a restricted set of allowed measurements in Quantum Mechanics. Furthermore, unsharp measurements often occur during the experimental realization of projective measurements of observables such as energy, position and spin. This is because projective measurements of these quantities are an idealization and do not take into account measurement error which may result in the actual measurement being unsharp.

A more general description of measurements was introduced to Quantum Theory in the 1970, cp. [4]. In this framework the statistics of any quantum mechanical measurement can be evaluated if the set of Effects, $\{E_a\}$, associated with the possible results, a, of the measurement as well as the state of the system immediately before the measurement, ρ_{in}, are known. The set of Effects forms a positive operator valued measure (POVM) [5] and each element E_a of the set corresponds to the event of "measuring a". Note that for projective measurements the Effects $\{E_a\}$ correspond to the projectors $\{P_a\}$. The probability for the result a to occur can be calculated as

$$p(a, \rho_{\text{in}}) = \text{Tr}[E_a \rho_{\text{in}}]. \tag{3}$$

Unlike for projective measurements, however, knowing the Effects is in general not enough to determine the state of the system after measurement; we further need to know the Kraus measurement operators M_k which make up the POVM elements E_a, i.e.

$$E_a = \sum_{k \in I_a} M_k^{\dagger} M_k \tag{4}$$

where I_a is an index set associated with the measurement result a. The change of the state due to the measurement is then given by

$$\rho_{\text{in}} \to \rho_{\text{out}} = \frac{1}{p(a, \rho_{\text{in}})} \sum_{k \in I_a} M_k \rho_{\text{in}} M_k^{\dagger}. \tag{5}$$

Generalized or POVM measurements play an important role in Quantum Theory since they allow us to investigate fundamental question such as the information gain and state disturbance trade-off. Furthermore they are useful in developing novel experimental schemes like the quantum optical measurement of the number of photons in a cavity or the measurement of the position of an atom in a standing light wave, cp. [5].

3. Unsharp measurements

A generalized measurement is called pure if to each measurement result a there corresponds exactly one Kraus operator M_a. This preserves the purity of states in the so-called selective regime of measurement; i.e. conditioned on a concrete measurement result, a pure state $|\psi\rangle$ is mapped in the course of a measurement onto a pure state: $|\psi\rangle \to \frac{M_a}{N}|\psi\rangle$ where $N \equiv \sqrt{\langle\psi|E_a|\psi\rangle}$ is the norm of the state which is the square of the probability to obtain the measurement result. The completeness relation then reads,

$$\sum_a M_a^\dagger M_a = \mathbb{1}. \tag{6}$$

A measurement is called unsharp if, in addition, not all the Effects $E_a = M_a^\dagger M_a$ are projectors but the Effects must be mutually commutating,

$$[E_a, E_b] := E_a E_b - E_b E_a = 0 \quad \text{for all measurement results } a, b. \tag{7}$$

This restriction is necessary because it allows one to distinguish which Effect was measured. An unsharp measurement thus has the same requirements as a projective measurement, except that not all the Effects are projectors. [1]

4. Scheme for implementing unsharp measurements

Now that the foundation has been laid, it is interesting to know how unsharp measurement can be implemented in a realistic experiment. Implementation schemes for unsharp measurements on trapped ions were recently demonstrated in [6]. We will briefly review one of these schemes here. The intricate details of the ion traps are neglected in order to keep the discussion focused and understandable.

One possible method for carrying out an unsharp measurement is through an ancilla system as shown in [6]. The basic idea is as follows. Conventional quantum logic operations for ion traps allow us to introduce weak entanglement (see below) between the target ion and an auxiliary ion. A projective measurement carried out on the auxiliary ion then leads to a realization of an unsharp measurement on the target ion.

We have two different species of ions trapped in the same ion trap so they share collective vibrational (or phonon) modes. One is the target ion which we prepare in a coherent superposition of its qubit levels

$$|\psi_t^{(0)}\rangle = c_1|g_t\rangle + c_2|e_t\rangle \tag{8}$$

where $|g_t\rangle$ and $|e_t\rangle$ are two hyperfine electronic ground levels and $|c_1|^2 + |c_2|^2 = 1$. We aim to measure the z component of its effective spin σ_z, unsharply. For this purpose we choose to implement two symmetric measurement operators

$$M_0 = \sqrt{p_0}|g_t\rangle\langle g_t| + \sqrt{1-p_0}|e_t\rangle\langle e_t|, \tag{9}$$

$$M_1 = \sqrt{1-p_0}|g_t\rangle\langle g_t| + \sqrt{p_0}|e_t\rangle\langle e_t|, \tag{10}$$

which satisfy the necessary requirements for an unsharp measurement mentioned in the preceding section. The target ion must also have a third metastable excited state $|r_t\rangle$, to assist with the unsharp measurement. The second ion is the auxiliary ion which is used for sympathetic cooling of the phonon modes as well as for implementing the unsharp measurement on the target ion. Since the ions are of different species, the laser used to manipulate one ion does not affect the other, thus the ions can be addressed independently.

[1] Please note, that we consider unsharp measurements to be pure, while in the literature unsharp measurements are just required to have commuting effects, cp. [5].

The state of the system of ions in the trap after the ground-state cooling on the auxiliary ion can be described by

$$|\psi\rangle = |\psi_t^{(0)}\rangle|\psi_a\rangle|0\rangle = (c_1|g_t\rangle + c_2|e_t\rangle)|\psi_a\rangle|0\rangle \tag{11}$$

where $|\psi_a\rangle$ is the state of the auxiliary ion and $|0\rangle$ is vibrational ground state of the ions (i.e. zero phonons). The measurement preparation proceeds by applying four laser pulses to the target ion. The first pulse is a carrier pulse on resonance with the transition $|g_t\rangle \to |r_t\rangle$ for the duration $t = (2/\Omega_1)\cos^{-1}(\sqrt{p_0})$ where Ω_1 is the Rabi frequency connecting $|g_t\rangle$ and $|r_t\rangle$, which results in the state,

$$|\psi_1\rangle = \Big[c_1\Big(\sqrt{p_0}|g_t\rangle + \sqrt{1-p_0}|r_t\rangle\Big) + c_2|e_t\rangle\Big]|\psi_a\rangle|0\rangle. \tag{12}$$

The second pulse is a red sideband pulse between $|g_t\rangle$ and $|r_t\rangle$, which induces the transitions $|r_t\rangle|n\rangle \to |g_t\rangle|n+1\rangle$ and $|g_t\rangle|n\rangle \to |r_t\rangle|n-1\rangle$ where n is the number of phonons. Hence the two-ion state becomes

$$|\psi_2\rangle = \Big[c_1|g_t\rangle\Big(\sqrt{p_0}|0\rangle + \sqrt{1-p_0}|1\rangle\Big) + c_2|e_t\rangle|0\rangle\Big]|\psi_a\rangle. \tag{13}$$

It is important to note that the component $|g_t\rangle|0\rangle$ is unaffected because the state $|r_t\rangle|-1\rangle$ does not exist. The third pulse is again a carrier pulse but this one is on resonance with the transition $|e_t\rangle \to |r_t\rangle$ for the duration $t = (2/\Omega_2)\cos^{-1}(\sqrt{1-p_0})$ where Ω_2 is the Rabi frequency connecting $|e_t\rangle$ and $|r_t\rangle$, which leads to the state,

$$|\psi_3\rangle = \Big[c_1|g_t\rangle\Big(\sqrt{p_0}|0\rangle + \sqrt{1-p_0}|1\rangle\Big) + c_2\Big(\sqrt{1-p_0}|e_t\rangle + \sqrt{p_0}|r_t\rangle\Big)|0\rangle\Big]|\psi_a\rangle. \tag{14}$$

Finally, a red sideband pulse between $|e_t\rangle$ and $|r_t\rangle$ yields,

$$|\psi_4\rangle = \Big[c_1|g_t\rangle\Big(\sqrt{p_0}|0\rangle + \sqrt{1-p_0}|1\rangle\Big) + c_2|e_t\rangle\Big(\sqrt{1-p_0}|0\rangle + \sqrt{p_0}|1\rangle\Big)\Big]|\psi_a\rangle, \tag{15}$$

$$= \Big[\Big(\sqrt{p_0}c_1|g_t\rangle + \sqrt{1-p_0}c_2|e_t\rangle\Big)|0\rangle + \Big(\sqrt{1-p_0}c_1|g_t\rangle + \sqrt{p_0}c_2|e_t\rangle\Big)|1\rangle\Big]|\psi_a\rangle. \tag{16}$$

Now, a projective measurement on the vibrational state in the basis $\{|0\rangle, |1\rangle\}$ results in the internal state of the target being either

$$\Big(\sqrt{p_0}c_1|g_t\rangle + \sqrt{1-p_0}c_2|e_t\rangle\Big) \tag{17}$$

or

$$\Big(\sqrt{1-p_0}c_1|g_t\rangle + \sqrt{p_0}c_2|e_t\rangle\Big). \tag{18}$$

The projection measurement of the vibrational state is accomplished through a Quantum Logic Spectroscopy measurement [7]. An unsharp measurement of σ_z is thus realized since this sequence is equivalent to applying the measurement operators of Eqs. (9) and (10) on the initial state of the target atom, i.e. Eq. (8).

5. Discussion

We introduced the concept of generalized measurements and reformulated the measurement postulate in terms of them. The properties of unsharp measurements where presented and an application of these measurements for a realistic experiment was demonstrated. Unsharp measurements are an appropriate means to obtain information about an observable under the constraint that the disturbance of a system must not exceed a certain level.

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