Stellar perturbation via Lie derivatives

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Abstract. Perturbation theory uses Lagrangian techniques that require vector fields to be compared at finitely separated points. This method can be generalised to the strong gravitational field regime in one of two ways, using either covariant or Lie derivatives. In this paper, I argue that those methods based on the Lie derivative are more useful. The Lie derivative provides a clear picture of how the deformation of the fluid flow takes place. It also provides a natural way to discuss large perturbations. I apply this method to some elementary stability problems in the study of stellar structure.

1. Introduction

Perturbation techniques in fluid dynamics were developed principally to study the stability of solutions of the equations of motion of a fluid [1]. In general, the study of stability requires only infinitesimal deviations from the known solution to be considered. In situations where a flow is known to be stable, the same techniques used in the study of stability can be applied to investigate the properties of flows that differ only by small amounts from a stable flow and to deduce approximate solutions for them.

In this paper, we consider the different available definitions of Lagrangian displacement. In Section 2, show why the elementary definition of it is unsuitable for use in advanced studies, discuss the alternative definition in terms of covariant derivatives, and show that the most general definition available is in terms of the Lie derivative. In Section 3, we outline a general mathematical framework for extending perturbation theory. We review the concepts of Lie transport and derivative in Section 4, and justify the definition of Lagrangian difference in terms of the Lie derivative in Section 5.

2. Lagrangian variation

Two types of variation are used in perturbation theory, called respectively Eulerian and Lagrangian [1]. The Eulerian description of fluids is in terms of time-dependent fields defined in three dimensional space. The Eulerian variation measures by how much a given field of the perturbed flow differs from the corresponding field in the unperturbed flow. To determine the Eulerian variation, one inspects the value of the field for the perturbed flow at a given point $\vec{x}$ in space at time t and compares it with the value of the corresponding field for the unperturbed flow *at the same space point* at time t. To express this analytically, denote the field for the perturbed and unperturbed flows generically by the symbols Q and Q_0 respectively. The fields

Q and Q_0 might be scalar, vector or tensor fields. Then the Eulerian difference for the field Q at position $\vec{x}$ at time t is defined to be

$$\delta Q(\vec{x},t) \quad = \quad Q(\vec{x},t) - Q_0(\vec{x},t) \tag{1}$$

The difference δQ is of two fields defined at the same space point, and so is well defined irrespective of the nature of the field Q. Elementary problems in perturbation theory can be solved exclusively in terms of δQ, which is easy to use mathematically, and is relatively easy to interpret physically. More advanced problems, however, require us to identify given material elements in the fluid and to track and compare their properties in the perturbed and unperturbed flows. In general, position $\vec{x}$ is occupied at time t in the two flows by *different* material elements of the fluid, so δQ cannot be used directly to compare how the variable Q differs in the two flows for one and the same material element.

To track a given fluid element, we need to switch from the Eulerian description of the fluid to the Lagrangian. In the Lagrangian description, each material point of the fluid is assigned, once and for all, a fixed set (a^1, a^2, a^3) of coordinates and its motion through space is the tracked by means of its trajectory

$$x^i \quad = \quad F^i(\vec{a},t) \tag{2}$$

The function $\vec{F}$ is called the *flow*. It delivers the position $\vec{x}$ in space of the particle at time t when its material coordinates $\vec{a}$ and the value of t are used as inputs into the function $\vec{F}$. Denote the perturbed and unperturbed flows by $\vec{F}$ and $\vec{F_0}$ respectively. The Lagrangian difference ΔQ in the value of a fluid variable Q is defined in elementary treatments by

$$\Delta Q \quad = \quad Q(\vec{F}(\vec{a},t),t) - Q_0(\vec{F_0}(\vec{a},t),t) \tag{3}$$

Using the same value of $\vec{a}$ in both arguments guarantees that we are inspecting the value of Q for the same material element in both flows.

This definition of ΔQ is problematic. It requires us to compare the values of the fields Q and Q_0 at *different* points in space. This poses no difficulties when Q and Q_0 are scalar fields. However, in general, vector and tensor fields such as fluid velocity, stress and rate of strain, at different space points cannot be compared. To effect such a comparison, we need a method for 'copying' the field Q to the same space point used to evaluate Q_0. We can then form the difference between the 'copy' of Q and Q_0 at the same space point $\vec{x} = F_0(\vec{a},t)$. This problem does not occur when the fluid flow takes place in an Euclidian space equipped with Cartesian coordinates, the setting normally assumed for perturbation problems. However, it rears its head the moment we choose to use non-Cartesian coordinates, or to consider flows in spaces other than Euclidian.

Tensor fields at different locations may be compared by one of two methods. The one used most commonly relies on the presence in the space of a connection and of an associated covariant derivative [2]. The 'remote' field Q can then be copied by parallel transport to the position where Q_0 is measured and compared to it. The advantage of using this method is that it provides the most obvious generalisation of the elementary definition of Lagrangian displacement, given in equation (3). Comparison of tensor fields at different points in an Euclidian space relies on the property of distant parallelism, which is a special property of its connection. This property is obscured when we use Cartesian coordinates, 'hidden in plain sight' as it were, but forces itself into our consciousness when we change to curvilinear coordinates, manifesting itself through the path independence of parallel transport. Distant parallelism means that we are able us to parallel-transport $Q(\vec{x}',t)$ unambiguously from position $\vec{x}'$ to position $\vec{x}$ and there to construct an unique 'copy' $\widetilde{Q}(\vec{x},t)$ of $Q(\vec{x}',t)$. Using this unique copy, we can then form the difference

$$\Delta Q \quad = \quad \widetilde{Q}(\vec{x},t) - Q_0(\vec{x},t) \tag{4}$$

The quantity ΔQ is unambiguously defined by virtue of the path independence of parallel transport in Euclidian space, and can thus be used to replace definition (3) of the Lagrangian difference. This definition leads to the approximate relation

$$\Delta Q \quad = \quad \delta Q + \nabla_{\vec{\xi}}\, Q \tag{5}$$

where $\vec{\xi}$ is the Lagrangian displacement (defined in the next section), used by many authors. See for example references [1] and [2].

Many spaces of interest, however, do not possess the property of distant parallelism, even when they possess a connection, so definition (4) of the Lagrangian difference also fails. In a general connected manifold, parallel transport is in general path dependent. The quantity $\widetilde{Q}(\vec{x}, t)$ is not uniquely defined, making definition (4) empty of content. The only exception to this occurs when the points $F(\vec{a}, t)$ and $F_0(\vec{a}, t)$ are infinitesimally close. This means that we can continue to use definition (4) provided that we are interested only in perturbed flows that differ only infinitesimally from the unperturbed one.

There is an alternative method available for comparing tensor fields at different positions. This method does not rely on the existence of a connection on the space, is available in all manifolds, and does not require the perturbation to be small. It is therefore suitable not only for the discussion of stability of flows in general spaces, but can also be extended to deal with large amplitude perturbations. In this method, the concept of Lie transport replaces that of parallel transport, and the Lie derivative that of the covariant derivative.

3. A model for perturbations

Rather than consider a single perturbed flow $\vec{F}$, which would force us to work only with finite differences in the flow variables, it is simpler to work with a one-parameter family $\vec{F}_\lambda$ of perturbed flows. One way to visualise these is by constructing a 1-dimensional continuum of spacetimes M_λ. We can then regard each perturbed flow F_λ as taking place on a separate sheet M_λ of these stacked spacetimes, with the flow on each sheet differing infinitesimally from one on the next sheet. Another way to visualise this is by regarding all of the flows as defined on a single spacetime M, which is the canonical projection of the spacetime stack in the direction of λ. This projection, applied to a single perturbed flow, is visualised by Lynden-Bell and Ostriker as a 'ghostly flow' accompanying the perturbed flow in much the same way as a residual image accompanies the real image on a poor quality television screen.

The 1-parameter family of perturbed flows can be regarded as 'distortions' of the unperturbed flow. Were we, at fixed time t, to allow the parameter λ to run through values from 0 to some finite value, each material element of the flow would trace out a curve in space. This three parameter family of curves defines the *distortion flow*, $G(\vec{x}, t, \lambda)$. This flow in turn defines a vector field $\vec{\eta}(\vec{x}, t, \lambda)$. Knowledge of $\vec{\eta}$ would allow us to deduce the distortion flow G by integration, and hence to construct fully the perturbed flow corresponding to any given parameter value λ. The problem of perturbation then reduces to that of finding equations that determine the distortion field $\vec{\eta}(\vec{x}, t, \lambda)$. The form and nature of these equations depend not only on the equations of motion of the fluid, but also on the constraints that the perturbations are assumed to obey.

The Lagrangian displacement field $\vec{\xi}(\vec{x}, t)$ of standard perturbation theory is related to the field $\vec{\eta}(\vec{x}, t, \lambda)$ at the parameter value $\lambda = 0$. If we confine ourselves to small perturbations, we can express the distortion flow approximately in the form

$$G^i(\vec{x}, t, \lambda) \quad = \quad G^i(\vec{x}, t, 0) + \lambda\, \frac{\partial G}{\partial \lambda}(\vec{x}, t, 0) + \cdots = x^i + \lambda \eta^i(\vec{x}, t, 0) \tag{6}$$

so that $\vec{\xi}(\vec{x}, t) = \lambda\, \vec{\eta}(\vec{x}, t, 0)$. We do not develop this general scheme further in this paper. We now return to the definition of the Lagrange difference in a general space.

4. The Lie derivative

Lie transport is the convective transport of objects by means of a flow [3]. Any point in the fluid will convect in time t to a new position in space. Any curve consisting of fluid points will also convect to a new position. A vector can always be constructed as the tangent to a given curve that passes through its point of attachment. The convected curve will have a tangent vector at the convective image of the attachment point. This vector is defined to be the convective image of the given vector and is said to be *Lie transported* by the flow. The Lie transport of a tensor is defined inductively from that of a vector by contracting it with the requisite number of vectors (and covectors) to form an invariant. The Lie transport of the tensor is then that tensor at the transported point that yields the same value of the invariant by contraction with the transported vectors as is yielded by the original tensor with the original vectors. If T is now taken to be a time independent tensor field rather than a single tensor, its Lie transported image can be compared at any point on a flow line with the tensor defined by the field at that point. The Lie derivative measures the rate at which the tensor of the field differs from its Lie transported version at the same point.

Formally, this may be seen as follows. Let $F(\vec{x}, t)$ be a flow. Denote by ϕ_t the diffeomorphims defined by F. Thus, $\phi_t(\vec{x}) = F(\vec{x}, t)$. The fluid point which at time $t = 0$ occupies position $\vec{x}$, will be found at time t at position $F(\vec{x}, t)$. The field T at this position has value $T(F(\vec{x}, t)) = T(\phi_t(\vec{x}))$, while at its original position it has value $T(\vec{x})$. Lie transport of $T(\vec{x})$ to position $F(\vec{x}, t)$ produces a tensor $\widetilde{T}$ at that position. It can be shown that $\widetilde{T}$ is given by $T\phi_t(T(\vec{x}))$, where $T\phi_t$ is the derivative of the mapping ϕ_t. Since this is a value at the point $F(\vec{x}, t)$, it can be compared directly to that of the field T at that point. The difference in these values is a well defined quantity, called the *Lie difference*. For small values of t, it can be used to define a differential quotient and, in the limit as $t \to 0$, it defines the *Lie derivative*. It is worth noting that we have not needed to introduce any additional structure on the space in the definition of any of these operations, and so this method of forming differences and derivatives is available on all types of manifold.

The above operations are easy to visualise physically. However, to get a cleaner mathematical definition, it is better to perform the above procedures in reverse. Instead of convecting $T(\vec{x})$ forwards with the flow to obtain its Lie transport $\widetilde{T}$, reverse the flow in time and Lie transport the value $T(\phi_t(\vec{x}))$ backwards to $\vec{x}$. Then compare this transported value with $T(\vec{x})$ as reference. Mathematically, the forward-transport and backward-transport procedures are equivalent and yield the same result. The time-reversed transport is called 'pull-back' and yields the field $T\phi_t^{-1}(T(\phi_t(\vec{x})))$. It is denoted more briefly by $\phi_t^* T(x)$. The Lie derivative is denoted by $\mathcal{L}_{\vec{X}} T$, where $\vec{X}$ is the generating vector field for the flow F (that is, the velocity field of the flow), and is thus defined by

$$\mathcal{L}_{\vec{X}} T(\vec{x}) = \lim_{\varepsilon \to 0} \frac{\phi_\varepsilon^* T(\vec{x}) - \phi_0^* T(\vec{x})}{\varepsilon} \tag{7}$$

Since ϕ_0 is the identity, we have $\phi_0^* T(\vec{x}) = T(\vec{x})$. All fields in the above definition are evaluated at the same point $\vec{x}$ in space, so we can omit arguments without fear of ambiguity to write

$$\mathcal{L}_{\vec{X}} T = \lim_{\varepsilon \to 0} \frac{\phi_\varepsilon^* T - T}{\varepsilon} \tag{8}$$

Note that the vector field $\vec{X}$ that generates the flow can be *any* vector field and need not be the velocity field of some actual fluid flow. For example, it could be the vector field $\vec{\xi}$ that generates perturbations or deformations of a given unperturbed flow.

5. Lagrangian difference

Suppose T_λ is a tensor field associated with the perturbed flow $F_\lambda(\vec{x}, t)$. We want to compare the value of T at at the fluid point which, in the unperturbed flow occupies position $\vec{x}$ at time t, with the value of the field T_λ at the position $G(\vec{x}, t, \lambda)$ occupied by the same fluid point at time t. In a general space, the only means by which this may be done is by Lie transporting the value $T_\lambda(G(\vec{x}, t, \lambda), t)$ backwards along the deformation flow G to position $\vec{x}$. This can be done via pull back and yields the value $TG_\lambda^{-1}(T_\lambda(G(\vec{x}, t, \lambda), t)) = G_\lambda^* T_\lambda(\vec{x}, t)$. This is the value of the tensor field $G_\lambda^* T_\lambda$ at $\vec{x}$, so it can be compared with the value $T_0(\vec{x}, t)$ of the field T_0. The difference

$$\Delta T = G_\lambda^* T_\lambda(\vec{x}, t) - T_0(\vec{x}, t) \tag{9}$$

is thus a well defined quantity. Since the 'classical' notion of a Lagrangian difference is undefined on a general manifold, and since the difference defined in the above equation encapsulates as closely as possible the spirit of the 'classical' notion of a Lagrangian difference, we may take this to be the correct generalisation of that concept to an arbitrary manifold [4].

To turn this definition into an useful mathematical operation, convert the above difference into a derivative. This is done as follows. First note that

$$\begin{aligned} \Delta T &= G_\lambda^* T_\lambda(\vec{x}, t) - G_0^* T_\lambda(\vec{x}, t) + G_0^* T_\lambda(\vec{x}, t) - T_0(\vec{x}, t) \\ &= \left[G_\lambda^* T_\lambda - T_\lambda\right](\vec{x}, t) + T_\lambda(\vec{x}, t) - T_0(\vec{x}, t) \end{aligned} \tag{10}$$

The last two terms are the Eulerian difference δT, while the first two are related to the Lie derivative. To first order in λ, we have

$$\Delta T = \lambda \, \mathcal{L}_{\vec{\eta}} T_\lambda(\vec{x}, t) + \delta T(\vec{x}, t) \tag{11}$$

Writing $\vec{\xi} = \lambda \vec{\eta}$ gives us the formula

$$\Delta T = \mathcal{L}_{\vec{\xi}} T_\lambda + \delta T \tag{12}$$

used by Friedman and Schutz. They advocate the use of this definition with the claim that it is 'somewhat more natural' than (5). They might have claimed more: this is the only definition of the Lagrangian difference that can be used universally, irrespective of the nature of the manifold on which the flow occurs. We define also the derivative

$$\frac{d}{d\lambda} T_\lambda \bigg|_{\lambda=0} (\vec{x}, t) = \lim_{\lambda \to 0} \frac{\Delta T}{\lambda} = \mathcal{L}_{\vec{\eta}} T(\vec{x}, t, 0) + \frac{\partial T}{\partial \lambda}(\vec{x}, t, 0) \tag{13}$$

which measures the rate at which the field T of the unperturbed flow changes per unit deformation as we move to the perturbed flow. Used in conjunction with the equations of motion of the fluid reformulated in terms of the Lie and exterior derivatives, the above scheme provides a method for the extension of perturbation techniques to encompass also large perturbations. This may provide an useful method for dealing with large amplitude oscillations in stars.

References

[1] Cox J P 1980 *Theory of Stellar Pulsation* Princeton University Press
[2] Lynden-Bell D and Ostriker J P 1967 *MNRAS* **136** 293
[3] Abraham R and Marsden J E 1985 *Foundations of Mechanics* Addison-Wesley Publishing Company
[4] Friedman J L and Schutz B 1978 *ApJ* **221** 937