The symmetry of Gauss's law and Ampère's law

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Abstract. The correct application of Gauss's law in electrostatics requires symmetry considerations which students often struggle with and which are often inadequately explained in textbooks. Ampere's law in magnetostatics is the analogue of Gauss's law, yet during its application symmetry arguments are seldom applied and other means are found to motivate the form of the magnetic field. A systematic approach for analyzing the symmetry necessary to apply Gauss's law is first presented. Then, with the important consideration that the symmetry applies to the magnetic force rather than the magnetic field, this approach is used for analyzing the symmetry necessary to apply Ampere's law. This strengthens the conceptual link between these two laws, but can also be used to introduce the interesting (but seldom discussed) idea that the magnetic field is an axial vector while the electric field is a polar vector.

1. Introduction

Several recent papers have highlighted the many difficulties that students struggle with when dealing with Ampere's law. Manogue *et al.* [1] have described Ampere's law and discussed an example problem in detail. They identified five skills required to solve it, the first of which is the ability to "recognize and use symmetry arguments". Analyses of students' responses to questions involving Ampere's law have shown that "There is particular difficulty involved with the symmetry aspect" [2], that "students were not really thinking about the … magnetic field when they set up their Amperian loops" [3] and that "It seems as if the student does not take into account the pattern of field" [4]. Manogue *et al.* [1] have suggested that the frustrations of students can be characterized by the sentence "I just don't know how to get started!" and they offer the excellent advice that "It helps right at the beginning of an Ampere's law problem to know both the direction of the magnetic field and the variables on which the magnitude depends." Nevertheless, student observations indicate that they do not use information about the magnetic field that is accessible to them [3]. It has been suggested that "The role of symmetry and the nature of a symmetry argument as used in E&M are not familiar to students" [2] and "… the argument is not so simple. Students and faculty alike can get themselves tied up in knots trying to argue which components add and which cancel" [1]. The difficulties involved are apparent from the fact that even the symmetry argument presented by Monague *et al.* [1] has been disputed [5].

A technique will be described to obtain the direction and functional dependence of the magnetic field based on symmetry arguments which are an extension of the method generally applied for establishing the form of the electric field when using Gauss's law. The parallel between Gauss's law of electrostatics and Ampere's law of magnetostatics is often pointed out and in fact they offer similar difficulties for student learning [4]. In principle, then, it may be an advantage to extend the symmetry arguments used for Gauss's law problems to make them applicable to Ampere's law problems, but this approach is not used in most textbooks and has not been referred in the recent papers already cited. In addition, the technique addresses the related problem that "students frequently fail to distinguish between forces and fields" [2]

2. Symmetry arguments for Gauss's law

Singh has pointed out that "Most textbooks do not sufficiently emphasize symmetry considerations or the chain of reasoning required to determine if Gauss's law is useful for calculating the electric field" [6]. Exceptions are the textbooks by, for example, Cook [7] and Wangsness [8] and since we shall be building on the symmetry concepts applicable for Gauss's law it is necessary to review them briefly.

Suppose we have an infinitely long straight cylindrical rod with uniform charge density and wish to find the form of the electric field. The geometry of the problem suggests the use of cylindrical coordinates, in which any electric field vector can be written most generally as

$$\mathbf{E} = E_s(s,\phi,z)\hat{\mathbf{s}} + E_\phi(s,\phi,z)\hat{\boldsymbol{\varphi}} + E_z(s,\phi,z)\hat{\mathbf{z}}\,. \tag{1}$$

Now a system can be described as having a symmetry if one can make some transformation that leaves the system unchanged. Firstly since no real change in the system occurs when the rod is shifted along its axis (changing z) or rotated about its axis (changing ϕ), the system has translation symmetry with respect to these coordinates and the electric field (and therefore all its components) cannot depend on z or ϕ: the general expression for the electric field reduces to

$$\mathbf{E} = E_s(s)\hat{\mathbf{s}} + E_\phi(s)\hat{\boldsymbol{\varphi}} + E_z(s)\hat{\mathbf{z}}\,. \tag{2}$$

Note that the s coordinate cannot be translated because it is associated with a "semi-axis" having a definite starting point (the axis of the cylinder). Secondly for those coordinates having translation symmetry (and which have consequently been removed as dependent variables for the electric field components) one can consider reversal symmetry: the change $z \to -z$ and therefore $\hat{\mathbf{z}} \to -\hat{\mathbf{z}}$ (which may be visualized as a reflection perpendicular to the rod's axis) means that if the electric field had an E_z component, then this would be reversed in the expression for the electric field despite no real change occurring in the system. Since this is impossible, E_z must be zero. In a similar way one may consider a reversal $\phi \to -\phi$ and hence $\hat{\boldsymbol{\varphi}} \to -\hat{\boldsymbol{\varphi}}$ to eliminate the possibility of an E_ϕ component, leaving the electric field with the form $\mathbf{E} = E_s(s)\hat{\mathbf{s}}$.

After applying this method several times for different geometries it becomes apparent that in Gauss's law problems the coordinates which have translation symmetry always also have reversal symmetry and so the components that are eliminated always correspond to the coordinates which have been eliminated. However, it is suggested that one should retain all the steps and distinguish between translation and reversal symmetry because this will become important in Ampere's law problems.

3. Symmetry arguments for Ampere's law

Now consider the Ampere's law problem posed by Monague *et al.* [1] of an infinitely long straight (uncharged) cylindrical wire carrying a current along its axis, with the current density proportional to the distance from the axis. This wire has a similar shape to the uniformly charged rod previously considered (although it is uncharged), but whereas the electric field of the charged rod had the form $\mathbf{E} = E_s(s)\hat{\mathbf{s}}$ one now has to justify why the magnetic field of the current carrying wire has the form $\mathbf{B} = B_\phi(s)\hat{\boldsymbol{\varphi}}$. Monague *et al.* [1] apply a symmetry argument to eliminate the B_s component but state "the only way we know to establish the lack of a parallel component is to use the Biot-Savart law". Their method has been criticised by de Wolf [5] who provides alternative arguments and shows that both components can be eliminated based on the equation $\oiint \mathbf{B}\cdot d\mathbf{a} = 0$. A different approach which we suggest is to introduce one important change in perspective that allows the symmetry arguments which are applied for Gauss's law problems to be carried over or extended to Ampere's law problems: the infinitely long wire suggests the use of cylindrical coordinates and we write the magnetic field vector in general as

$$\mathbf{B} = B_s(s,\phi,z)\hat{\mathbf{s}} + B_\phi(s,\phi,z)\hat{\boldsymbol{\phi}} + B_z(s,\phi,z)\hat{\mathbf{z}} \,. \tag{3}$$

Firstly no real change occurs in the system when the wire is shifted along its axis (changing z) or rotated about its axis (changing ϕ), so the magnetic field components cannot depend on z or ϕ (translation symmetry) and the general expression for the magnetic field reduces to

$$\mathbf{B} = B_s(s)\hat{\mathbf{s}} + B_\phi(s)\hat{\boldsymbol{\phi}} + B_z(s)\hat{\mathbf{z}} \,. \tag{4}$$

This expression is given by Cook [9], who then abandons the symmetry approach. Taking it further, we secondly consider reversal symmetry for the coordinates which we have already found to have translation symmetry: a reversal $\phi \to -\phi$ and $\hat{\boldsymbol{\phi}} \to -\hat{\boldsymbol{\phi}}$ leaves the system unchanged and therefore eliminates the possibility of a B_ϕ component, but the change $z \to -z$ and therefore $\hat{\mathbf{z}} \to -\hat{\mathbf{z}}$ (which may be visualized as a reflection perpendicular to the wire's axis) does not leave the system unchanged since the current is reversed and now flows in the opposite direction. Therefore one is left with $\mathbf{B} = B_s(s)\hat{\mathbf{s}} + B_z(s)\hat{\mathbf{z}}$ instead of the expected $\mathbf{B} = B_\phi(s)\hat{\boldsymbol{\phi}}$, indicating that there must be some flaw in the reasoning, which is satisfying to resolve.

The fault is a failure to distinguish between forces and fields: neither the electric field nor the magnetic field are directly measurable, but can be considered through the effects of their respective forces $\mathbf{F} = q\mathbf{E}$ and $\mathbf{F} = q\mathbf{v}\times\mathbf{B}$. Therefore the symmetry of the system should not be considered to apply to the fields, but rather to possible forces. In the electric case the field and force are parallel and one is not led to the wrong conclusion, but in the magnetic case the distinction is vital since the field is perpendicular to the force due to the cross product. Adjusting our previously reasoning to apply to the magnetic force instead of the magnetic field, we have actually obtained

$$\mathbf{F} = F_s(s)\hat{\mathbf{s}} + F_z(s)\hat{\mathbf{z}} \,. \tag{5}$$

Considering that the velocity in $\mathbf{F} = q\mathbf{v}\times\mathbf{B}$ could be in any direction, the magnetic field must be in the $\hat{\boldsymbol{\phi}}$ direction to ensure it is perpendicular to the force i.e., $\mathbf{B} = B_\phi(s)\hat{\boldsymbol{\phi}}$ as required. It is interesting to note that the translational symmetry along the axis ($\hat{\mathbf{z}}$ direction) may be relaxed without influencing the direction of the magnetic field: if this is done one gets the magnetic force as $\mathbf{F} = F_s(s,z)\hat{\mathbf{s}} + F_z(s,z)\hat{\mathbf{z}}$ and the magnetic field as $\mathbf{B} = B_\phi(s,z)\hat{\boldsymbol{\phi}}$ which is applicable to the toroid. This approach for finding the form of the magnetic field is also easily applied in the case of solenoids (both circular or with arbitrary cross-section) and uniform flat sheets of current.

4. Conclusion

An instructive change in perspective allows the method that is used to analyze the symmetry in Gauss's law problems to be extended to Ampere's law problems in order to find the form of the magnetic field without reference to the Biot-Savart law or other laws of electromagnetism (besides the Lorentz force law). The magnetic field direction is obtained as the direction perpendicular to the plane to which the magnetic force is constrained by symmetry.

There is some ambiguity here, for given a plane one may choose two opposite directions both perpendicular to it. Symmetry cannot determine which one is correct and indeed, because the magnetic field is an axial (or pseudo-) vector rather than a polar vector (like the electric field) the correct choice is purely conventional, based on the right hand rule [10]. Thus the symmetry considerations are not only directly applicable to Ampere's law, but in applying them significant insight into the nature of the magnetic field can also be obtained.

5. References

[1] Manogue CA, Browne K, Dray T and Edwards B 2006 *Am. J. Phys.* **74** 344

[2] Guisasola J, Almudí JM and Zuza K 2010 *Am. J. Phys.* **78** 1207

[3] Wallace CS and Chasteen SV 2010 *Phys. Rev. ST Physics Ed. Research* **6** 020115

[4] Guisasola J, Almudí JM, Salinas J, Zuza K and Ceberio M 2008 *Eur. J. Phys.* **29** 1005

[5] de Wolf DA 2007 *IEEE Trans. Education* **50** 256

[6] Singh C 2006 *Am. J. Phys.* **74** 923

[7] Cook DM 2002 *The theory of the electromagnetic field* (Mineola: Dover Publications) p. 105

[8] Wangsness RK 1986 *Electromagnetic fields,* 2nd ed. (New York: John Wiley and Sons) pp. 60-63

[9] Ref. 7, p. 143.

[10] Jackson JD 1999 *Classical Electrodynamics,* 3rd ed. (New York: John Wiley and Sons) p. 271