Anomalous Dimensions of Heavy Operators from Magnon Energies

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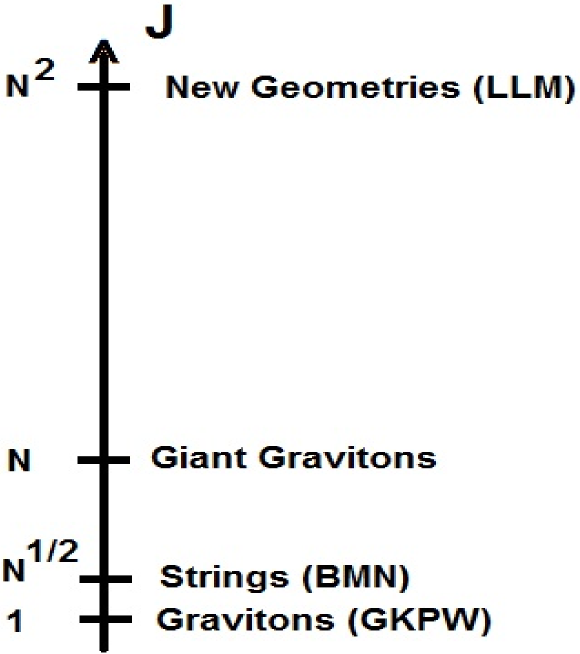
Spectrum of anomalous dimensions in the planar limit of $\mathcal{N} = 4$ super Yang-Mills theory is solved - thanks to integrability.

In this talk we will study the spectrum of anomalous dimensions of heavy operators (with a bare dimension of order N) both in the gauge theory and in the dual gravity theory.

This large N but non-planar limit is less understood than the planar limit - but also seems well worth study!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N^2
New Geometries (LLM)
N
Giant Gravitons
N^1/2
Strings (BMN)
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Gravitons (GKPW)

What are the key differences between large N but non-planar limits and the planar limit?

Distinct multi-trace structures are orthogonal in the planar limit.

$$\langle O_{\text{structure 1}} O_{\text{structure 2}} \rangle \propto \delta_{\text{structure 1;structure 2}}$$

$$\langle \frac{\mathrm{Tr}(Z^{J_1})}{\sqrt{J_1 N^{J_1}}} \frac{\mathrm{Tr}(Z^{J_2})}{\sqrt{J_2 N^{J_2}}} \frac{\mathrm{Tr}(Z^{\dagger\, J_3})}{\sqrt{J_3 N^{J_3}}} \rangle = \frac{\sqrt{J_1 J_2 J_3}}{N} \delta_{J_1+J_2;J_3}$$

Orthogonality breaks down at $J_i \sim N^{\frac{2}{3}}$ (rough estimate) $\Rightarrow$ different trace structures

[Balasubramanian, Berkooz, Naqvi, Strassler, hep-th/0107119]

This spoils the integrability found in the planar limit of $\mathcal{N} = 4$ super Yang-Mills theory.

Key Idea: map the dilatation operator into the Hamiltonian of an integrable spin chain by identifying single trace operators with states of the spin chain.

Crucially uses the fact that distinct operator-trace structures don't mix $\Rightarrow$ dilatation operator doesn't take you out of the space of single traces.

Two more important differences:

Not all operators are independent: trace relations; example for $N = 2$:

$$\mathrm{Tr}(Z)^3 - 3\mathrm{Tr}(Z^2)\mathrm{Tr}(Z) + 2\mathrm{Tr}(Z^3) = 0$$

Non-planar diagrams must be summed.

These issues can be effectively handled using an approach based on the symmetric group.

Permutations provide the natural language to describe this sector of the theory.

$$\mathrm{Tr}Z\mathrm{Tr}Z = Z^{i_1}_{i_1} Z^{i_2}_{i_2} \qquad \mathrm{Tr}(Z^2) = Z_{i_2}{}^{i_1} Z^{i_2}_{i_1}$$

Lower labels permuted with respect to upper labels.

$$\mathrm{Tr}Z\mathrm{Tr}Z = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \equiv \mathrm{Tr}(\sigma Z^{\otimes 2}) \quad \sigma = (1)(2)$$

$$\mathrm{Tr}(Z^2) = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \equiv \mathrm{Tr}(\sigma Z^{\otimes 2}) \quad \sigma = (12)$$

Language for arbitrary multitrace operators

$$\mathrm{Tr}(\sigma Z^{\otimes n}) = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \cdots Z^{i_n}_{i_{\sigma(n)}}$$

Any multitrace operator composed from k fields corresponds to a $\sigma \in S_k$.

Permutations in the same conjugacy class determine the same operator.

Is this a useful description?

$$\langle Z^i{}_j (Z^\dagger)^k{}_l \rangle = \delta^i_l \delta^k_j$$

$$\langle\, A^{i_1 \cdots i_n}_{j_1 \cdots j_n} Z^{j_1}_{i_1} \cdots Z^{j_n}_{i_n} \;\; B^{k_1 \cdots k_n}_{l_1 \cdots l_n} (Z^\dagger)^{l_1}_{k_1} \cdots (Z^\dagger)^{l_n}_{k_n} \,\rangle$$

$$= \sum_{\sigma \in S_n} \mathrm{Tr}(A \sigma B \sigma^{-1})$$

Summing over all permutations is a sum over all ribbon graphs.

$$\langle Z^i{}_j (Z^\dagger)^k{}_l \rangle = \delta^i_l \delta^k_j$$

$$\langle\, A^{i_1\cdots i_n}_{j_1\cdots j_n} Z^{j_1}_{i_1} \cdots Z^{j_n}_{i_n} \;\; B^{k_1\cdots k_n}_{l_1\cdots l_n} (Z^\dagger)^{l_1}_{k_1} \cdots (Z^\dagger)^{l_n}_{k_n} \,\rangle$$

$$= \sum_{\sigma\in S_n} \mathrm{Tr}(A\sigma B\sigma^{-1})$$

Projection operators obey

$$[P_A, \sigma] = 0 \qquad P_A P_B = \delta_{AB} P_A$$

Thus the sum over all ribbon graphs is

$$\sum_{\sigma\in S_n} \mathrm{Tr}(P_A \sigma P_B \sigma^{-1}) = n!\delta_{AB} \mathrm{Tr}(P_A) = n!\delta_{AB} d_A$$

$$\chi_R(Z) \propto (P_R)^{i_1\, i_2 \cdots\, i_n}_{j_1\, j_2 \cdots\, j_n} Z^{j_1}_{i_1} Z^{j_2}_{i_2} \cdots Z^{j_n}_{i_n}$$

$$\chi_R(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \mathrm{Tr}(\sigma Z^{\otimes n})$$

R specifies an irrep of S_n. $\chi_R(\sigma)$ is the character of σ in irrep R.

The Schur polynomials provide a basis for local operators built from Z.

[Corley, Jevicki, Ramgoolam, hep-th/0111222, Corley, Ramgoolam, hep-th/0205221]

Can these results be generalized to describe more than one matrix?

Restricted Schur polynomial

$$\chi_{R,(r,s)\alpha\beta}(Z,Y) =$$

$$\frac{1}{n!m!}\sum_{\sigma\in S_{n+m}} \mathrm{Tr}_{(r,s)\alpha\beta}(\Gamma^R(\sigma))\mathrm{Tr}(\sigma Z^{\otimes n}\otimes Y^{\otimes m})$$

R is an irrep of S_{n+m}. (r,s) is an irrep of $S_n \times S_m$. α, β are multiplicity labels.

[Balasubramanian, Berenstein, Feng, Huang hep-th/0411205; Bhattacharyya, Collins, dMK arXiv:0801.2061]

Restricted Schur polynomials define a basis.

$$\langle \chi_{R,(r,s)\mu\nu}(Z,Y)\chi_{S,(t,u)\alpha\beta}(Z,Y)^\dagger \rangle$$
$$= N(R,r,s)\delta_{RS}\delta_{rt}\delta_{su}\delta_{\mu\alpha}\delta_{\nu\beta}$$

$$\mathrm{Tr}(\sigma Z^{\otimes n} Y^{\otimes m})$$
$$= \sum_{R,(r,s)\alpha\beta} \mathrm{Tr}_{(r,s)\beta\alpha}(\Gamma^R(\sigma))\chi_{R,(r,s)\beta\alpha}(Z,Y)$$

[Bhattacharyya, Collins, dMK arXiv:0801.2061, Bhattacharyya, dMK, Stephanou arXiv:0805.3025. See also Brown, Heslop, Ramgoolam arXiv0711:0176, Kimura, Ramgoolam arXiv:0709.2158]

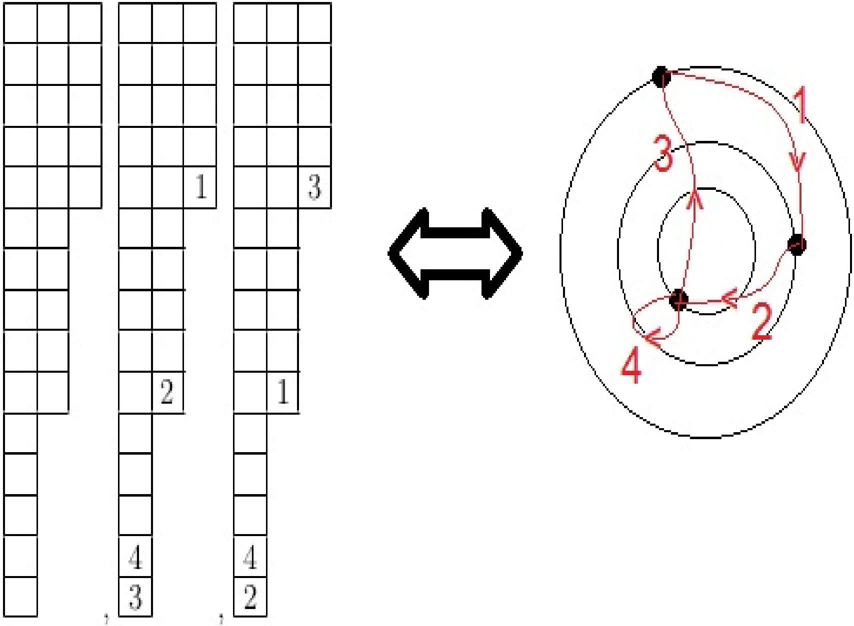
$$O(\{k_i\}) = \frac{1}{n!} \sum_{\sigma \in S_{n+1}} \chi_{R,R^1}(\sigma) Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_n}_{i_{\sigma(n)}} W^{i_{n+1}}_{i_{\sigma(n+1)}}$$

$$W^i_j = \left(YZ^{k_1} YZ^{k_2-k_1} Y \cdots YZ^{k_{L_I}-k_{L_I-1}} Y\right)^i_j$$

For large N correlators: sum all Z contractions but only planar W contractions!

$$\leftrightarrow \left| n; \{\, k_1, k_2, \cdots k_L \,\} \right\rangle$$

[dMK, Smolic, Smolic hep-th/0701066,0701067, Bekker, dMK, Stephanou arXiv:0710.5372]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What happens when we move beyond the free theory?

What is the action of the dilatation operator?

$$D = -g_{YM}^2 \mathrm{Tr}\left(\left[Z, Y\right]\left[\frac{d}{dZ}, \frac{d}{dY}\right]\right)$$

[Beisert, Kristjansen, Staudacher, hep-th/0303060]

Mixing is highly constrained: at L-loops at most L boxes in the Young diagram labeling the operator can change.

Rather simple expressions in terms of the factors of the Young diagram.

[Bekker, dMK, Stephanou arXiv:0710.5372, De Comarmond, dMK, Jefferies arXiv:1004.1108]

N	N+1	N+2	N+3
N-1	N		
N-2	N-1		
N-3			

$$|\text{giant momentum}; \{\text{magnon positions}\}\rangle$$

Acting on the bulk magnon:

$$D|n;\{x_1, x_2, x_3\}\rangle = g_{YM}^2\Big[2|n;\{x_1, x_2, x_3\}\rangle$$

$$-|n;\{x_1, x_2 - 1, x_3\}\rangle - |n;\{x_1, x_2 + 1, x_3\}\rangle\Big]$$

$$\underbrace{ZZZ\cdots Z}\overbrace{Y Z^{x_2 - x_1} Y Z^{x_3 - x_2} Y}\underbrace{Z\cdots Z}$$

$$|\text{giant momentum}; \{\text{magnon positions}\}\rangle$$

Acting on a boundary magnon:

$$D|n; \{x_1, x_2, x_3\}\rangle = g_{YM}^2 \Big[\left(1 + \frac{c}{N}\right) |n; \{x_1, x_2, x_3\}\rangle$$

$$- \sqrt{\frac{c}{N}} \left(|n; \{x_1 - 1, x_2, x_3\}\rangle + |n; \{x_1 + 1, x_2, x_3\}\rangle\right) \Big]$$

c is the factor of the box associated with the open string.

Eigenstate:

$$= \sum_{m_1=0}^{J-1} \sum_{m_2=0}^{m_1} q_1^{m_1} q_2^{m_2} |n+m_1-m_2; \{J-m_1+m_2\}\rangle$$

$$+ \sum_{m_2=0}^{J-1} \sum_{m_1=0}^{m_2} q_1^{m_1} q_2^{m_2} |n+J+m_1-m_2; \{m_2-m_1\}\rangle$$

Zero momentum constraint: $q_1 = q_2^{-1}$

For a giant graviton with momentum n, we find for a boundary magnon

$$E_1 = g^2 \left(1 + \left[1 - \frac{n}{N} \right] - \sqrt{1 - \frac{n}{N}} (q_1 + q_1^{-1}) \right)$$

and for a bulk magnon

$$E_2 = g^2 \left(2 - q_2 - q_2^{-1} \right)$$

The eigenstates enjoy an $su(2|2)^2$ symmetry.

Each magnon transforms in a centrally extended $su(2|2)^2$ representation. The momentum of each magnon determines the central charge of its representation.

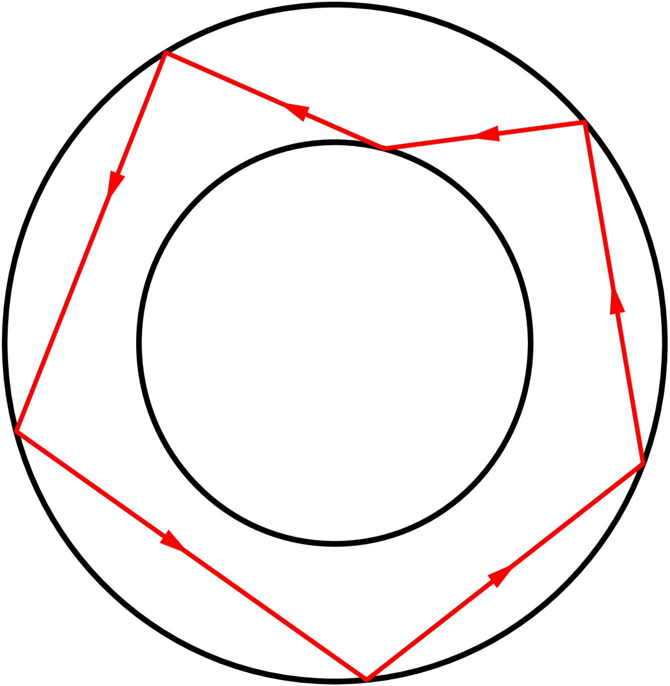
[Beisert, hep-th/0511082, nlin/0610017]

The zero momentum constraint ensures that the central extension of the eigenstate vanishes.

The dual string solution

The dual string solution

$$\underbrace{ZZZ\dots Z}\overbrace{Y\,Z^{x_2-x_1}\,Y\,Z^{x_3-x_2}\,Y}\underbrace{Z\dots Z}$$

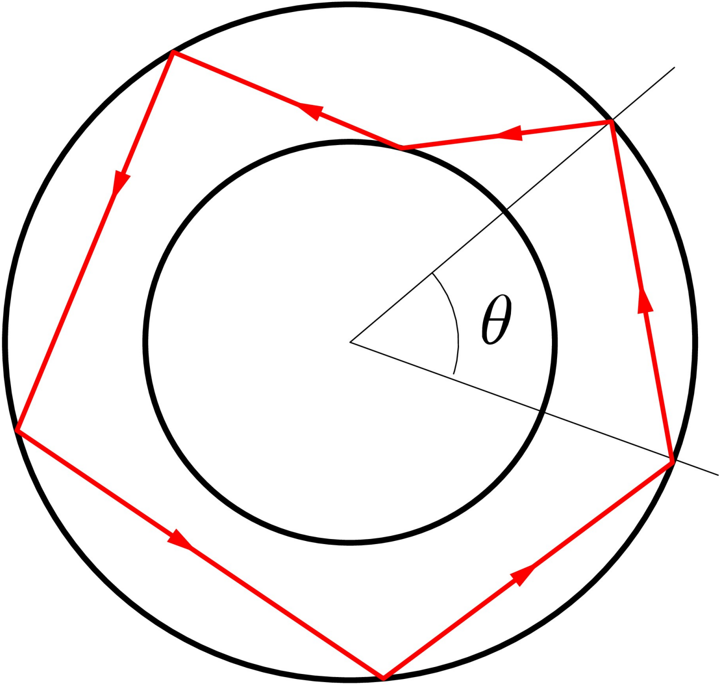


Each red segment is a magnon.

The $su(2|2)$ central charges are given geometrically.

$$E = \sqrt{1 + 2\lambda |k|^2} = 1 + \lambda |k|^2 + \cdots$$

[Berenstein, Correa, Vazquez hep-th/0509015, Maldacena, Hofman hep-th/0604135, arXiv:0708.2272]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$$
\begin{aligned}
E &= 1 + 4\lambda \sin^2 \frac{\theta}{2} + O(\lambda^2) \\
&= 1 + \lambda(2 - e^{i\theta} - e^{-i\theta}) + O(\lambda^2) \\
&= 1 + \lambda(2 - q - q^{-1}) + O(\lambda^2)
\end{aligned}
$$

[Berenstein, Correa, Vazquez hep-th/0509015, Maldacena, Hofman hep-th/0604135, arXiv:0708.2272]

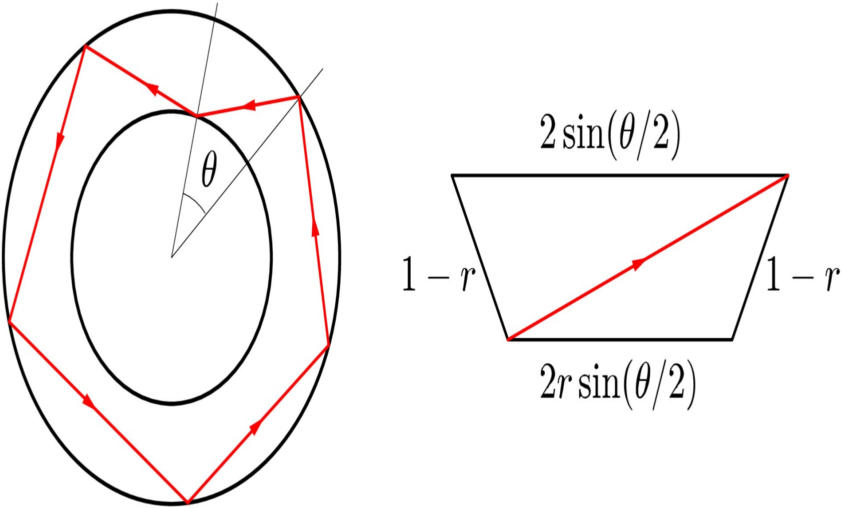


Figure:

$$E = 1 + \lambda((1 - r)^2 + 4r\sin^2\frac{\theta}{2}) + O(\lambda^2)$$

$$= 1 + \lambda\left(1 + r^2 - r(e^{i\theta} + e^{-i\theta})\right) + O(\lambda^2)$$

$$= 1 + \lambda\left(1 + 1 - \frac{n}{N} - \sqrt{1 - \frac{n}{N}}(q + q^{-1})\right)$$
$$+O(\lambda^2)$$

Can compute $su(2|2)$ invariant magnon scattering matrix for scattering of bulk and boundary magnons.

Results agrees with weak coupling.

Central charges of the total state must be preserved $\Rightarrow$ scattering is not elastic; not integrable

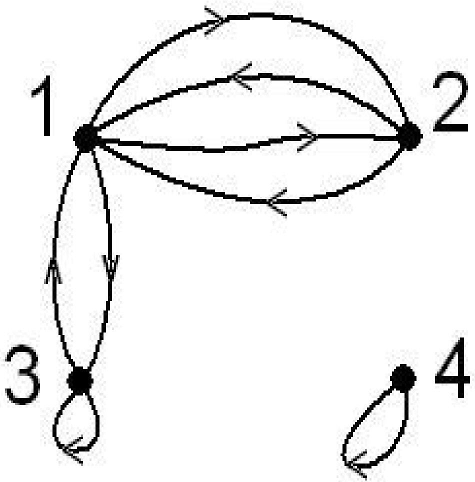
$$\chi_{R,(r,s)\alpha\beta}(Z,Y) =$$

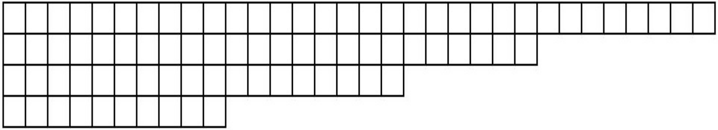
$$\frac{1}{n!m!}\sum_{\sigma\in S_{n+m}} \mathrm{Tr}_{(r,s)\alpha\beta}(\Gamma^R(\sigma))\mathrm{Tr}(\sigma Z^{\otimes n}\otimes Y^{\otimes m})$$

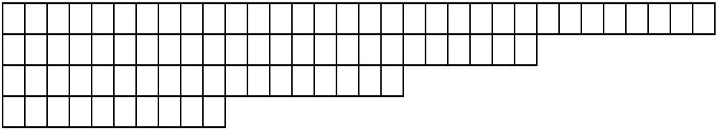
$$DO_{R,r}(\sigma) = -\frac{g_{YM}^2}{8\pi^2}\sum_{i<j} n_{ij}(\sigma)\Delta_{ij}O_{R,r}(\sigma)\,.$$

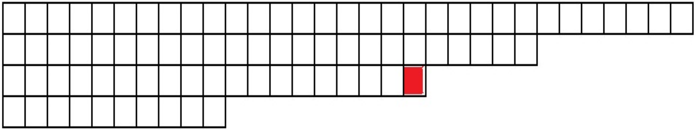
[dMK, Dessein, Giataganas, Mathwin arXiv:1108.2761, dMK, Ramgoolam arXiv:1204.2153]

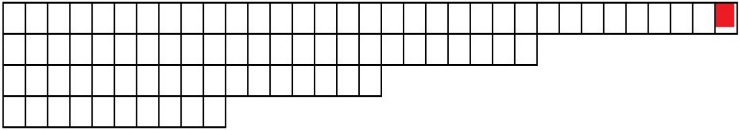
$$DO(b_0, b_1, b_2, b_3) = -g_{YM}^2(4\Delta_{12}+2\Delta_{13})O(b_0, b_1, b_2, b_3)$$



Δ_{13}  $=$

$(\alpha+\beta)$ 

$-\sqrt{\alpha\beta}$ 

$-\sqrt{\alpha\beta}$ 

$$
\begin{aligned}
&\frac{g_{YM}^2 N}{8\pi^2}\left[\sqrt{1-\frac{c_1}{N}}-\sqrt{1-\frac{c_2}{N}}\right]^2 O(c_1, c_2, c_3)\\
&+\frac{g_{YM}^2 N}{8\pi^2}\left[\sqrt{1-\frac{c_2}{N}}-\sqrt{1-\frac{c_3}{N}}\right]^2 O(c_1, c_2, c_3)\\
&+\frac{g_{YM}^2 N}{8\pi^2}\left[\sqrt{1-\frac{c_3}{N}}-\sqrt{1-\frac{c_1}{N}}\right]^2 O(c_1, c_2, c_3)\\
&=\gamma O(c_1, c_2, c_3)
\end{aligned}
$$

$$D = -g_{YM}^2 \sum_{i<j} n_{ij}(\sigma) \times$$

$$\times \left[\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right)^2 - \frac{(x_i - x_j)^2}{4} \right]$$

Conclusions

Combinatorics of summing Feynman diagrams and constructing bases of local operators is solved using group representation theory approach.

Physics of excited giant gravitons ripe for exploration using gauge/gravity duality.

Many of the lessons/tools that worked in the planar limit are useful here too!

Thanks for your attention!