Unravelling of Open Quantum Random Walks

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Abstract. Recently, in joint work of S. Attal, C. Sabot, F. Petruccione and I. Sinayskiy the concept of Open Quantum Random Walks was introduced, by taking into account dissipation and decoherence that occur in open quantum systems. Open quantum random walks are formulated in terms of discrete completely positive maps for the density matrix. These walks are simulated efficiently with the help of quantum trajectories. Here we report the unravelling of some simple open quantum random walk.

1. Introduction

It is well known that random walks are a very useful mathematical tool, which found successful applications in physics, computer science, economics and biology. The Unitary Quantum Random Walk (UQRW) [1, 2] generalized this concept to the quantum domain [3] and widely applied in quantum computing [4]. Recently, experimental realizations of UQRW have been reported [5, 6, 7, 8]. For the last few years attempts were made to take into account decoherence and dissipation in the quantum walks formalism [9]. But, in all these approaches decoherence and dissipation is treated as a modification of the unitary quantum walk scheme, the destructive influence of which needs to be minimized and eliminated. The general framework of quantum stochastic walks was proposed [10], which incorporates unitary and non-unitary effects of the quantum Markovian dynamics.

Recently, a formalism for discrete time quantum walks was introduced [11], that naturally includes the behavior of open quantum systems. The formalism suggested is similar to the formalism of quantum Markov chains [12] and rests upon the implementation of appropriate completely positive maps [13].

In this paper, we will briefly review the concept of the OQRWs and for a particular example we will show the simulation of an OQRW in terms of quantum trajectories.

2. Open Quantum Random Walks

The dynamics of a walker with internal degrees of freedom in the Hilbert space $\mathcal{H}$ on the nodes (i, j) (i, j are elements of a finite or infinite countable set) of a graph is defined in terms of operators $B^i_j \in \mathcal{H}$. These operators describe the transformation in the internal degree of freedom of the walker due to the jump from node j to node i. The basic idea of the OQRW is to assume that for each j

$$\sum_i B^{i\dagger}_j B^i_j = I,$$

where $B_i^j \in \mathcal{H}$ and I denotes the identity in the appropriate space. This constraint has to be understood in the following way: the sum of all the effects leaving site j is I. The Hilbert space of states specified by the set of nodes will be denoted by $\mathcal{K}$ and will be assumed to have a basis $|i\rangle$. Obviously, this construction is a natural generalization of the "classical" Markov chain concept. In order to describe not only the change in the internal degrees of freedom of the walker, but also the transitions from node j to node i it is convenient to introduce the operators $M_j^i = B_j^i \otimes |i\rangle\langle j|$, which satisfy $\sum_{i,j} M_j^{i\dagger} M_j^i = I$. The OQRW can now be defined in terms of the following completely positive map on $\mathcal{H} \otimes \mathcal{K}$

$$\mathcal{M}(\rho) = \sum_i \sum_j M_j^i \, \rho \, M_j^{i\dagger}.$$

If we assume the initial density matrix of the system to be of the form

$$\rho = \sum_i \rho_i \otimes |i\rangle\langle i|,$$

with $\sum_i \mathrm{Tr}\rho_i = 1$, the iteration formula for the OQRW from step n to step $n+1$ can be expressed as

$$\rho^{[n+1]} = \mathcal{M}(\rho^{[n]}) = \sum_i \rho_i^{[n+1]} \otimes |i\rangle\langle i|,$$

where $\rho_i^{[n+1]} = \sum_j B_j^i \rho_j^{[n]} B_j^{i\dagger}$.

3. Unravelling of the OQRW in terms of Quantum Trajectories

To introduce the quantum trajectories formalism we start by considering a particular case of initial state of the system, namely, a walker that is localized at a single site,

$$\rho_0 = \rho \otimes |i\rangle\langle i|.$$

After one step the state of the walker will be given by

$$\rho_1 = \sum_j (B_i^j \rho B_i^{j\dagger}) \otimes |j\rangle\langle j| \, .$$

The probability to find the walker at the site j is given by $p_j = \mathrm{tr}(B_i^j \rho B_i^{j\dagger})$. If one performs measurements of the position of the walker at site j the state of the walker reads

$$\frac{1}{p_j} (B_i^j \rho B_i^{j\dagger}) \otimes |j\rangle\langle j| \, .$$

Repetition of this procedure gives rise to a classical Markov chain, valued in the set of states of the form $\rho \otimes |i\rangle\langle i|$. One can see that this procedure on average will simulate a master equation driven by $\mathcal{M}$:

$$\mathcal{E}[\rho_{n+1}] = \sum_j p_j \frac{1}{p_j} (B_i^j \rho_n B_i^{j\dagger}) \otimes |j\rangle\langle j| = \sum_j (B_i^j \rho_n B_i^{j\dagger}) \otimes |j\rangle\langle j| = \mathcal{M}(\rho_n) \, .$$

In particular, if the initial state of the system is the pure state $\rho = |\phi\rangle\langle\phi| \otimes |i\rangle\langle i|$, then the system remains in a pure state. It is easy to see that, any initial pure state $|\phi\rangle \otimes |i\rangle$ will jump randomly to one of the states

$$\frac{1}{\sqrt{p_i^j}} B_i^j |\phi\rangle \otimes |j\rangle$$

with probability

$$p_i^j = ||B_i^j|\phi\rangle||^2 .$$

We have a classical Markov chain valued in the space of wave functions of the form $|\phi\rangle \otimes |i\rangle$. On average, this random walk simulates the master equation driven by $\mathcal{M}$.

As illustration of this unravelling we consider an open quantum random walk on a line. In this case the transition matrices B_i^j are chosen to be,

$$B_i^{i-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}, \quad B_i^{i+1} = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix},$$

and the initial state of the walker is

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle) \otimes |0\rangle.$$

The results of the simulations are presented in figure 1. Figure 1(a),(b) and (c) we show three different quantum trajectories. One can clearly see the different qualitative character of some of them. The average over 1000 realizations is shown in the figure 1(d).

More work is in progress in applying the open quantum random walk formalism to quantum computing and quantum state transfer problems.

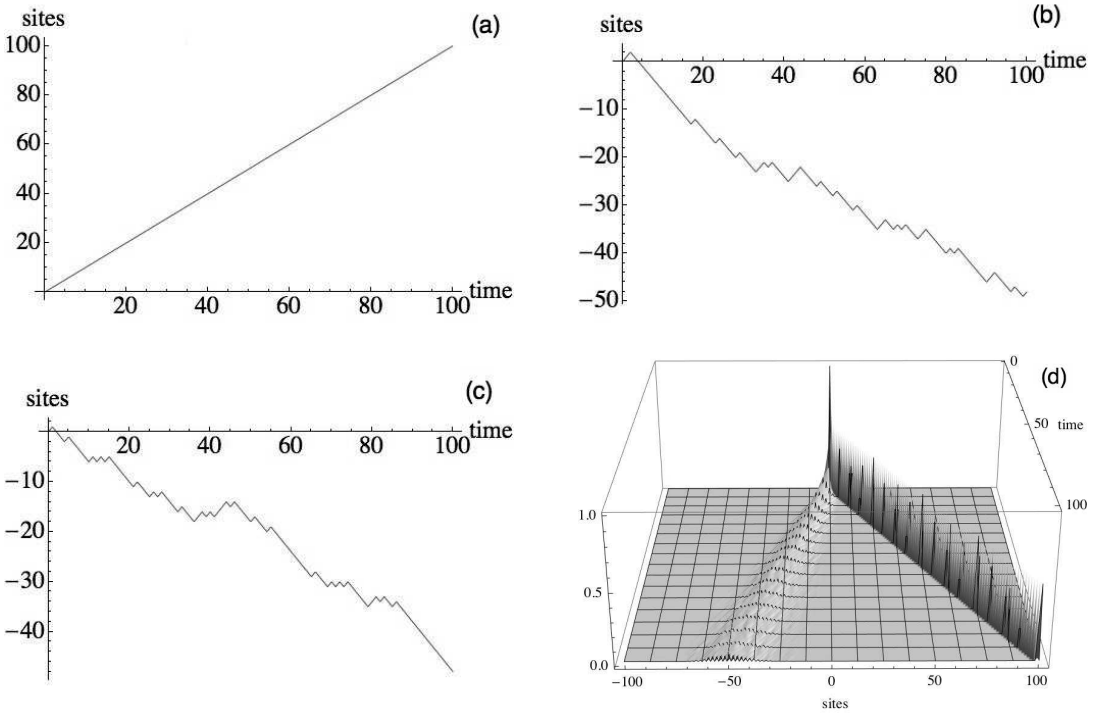

Figure 1. Simulation of the OQRW in term of quantum trajectories. The examples of the trajectories are shown on the curves (a)-(c), curve shows (d) average over 1000 trajectories

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