Efficiency of Open Quantum Walk implementation of the Dissipative Quantum Computing

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Abstract. A new type of quantum walk, exclusively based on dissipative dynamics is presented. An application of this open quantum walk for dissipative quantum computing is suggested. The approach is illustrated with the example of the phase estimation algorithm. It is explicitly demonstrated that open quantum walk based algorithms are more efficient than the traditional dissipative quantum computing approach. In particular, the open quantum walks can be designed to converge faster to the desired steady state and to increase the probability of detection of the outcome of the computation.

In the description of experimentally realizable quantum systems one should always include the unavoidable effect of the interaction with a dissipative and decoherent environment [1]. For most applications, the influence of decoherence and dissipation on the reduced systems needs to be eliminated or at least minimized. However, it was shown recently that the interaction with the environment not only can create complex entangled states [2, 3, 4, 5, 6, 7], but also allows for universal quantum computation [8]. Recently, a framework for discrete time open quantum walks (OQW) on graphs was proposed [9, 10], which is based upon exclusively dissipative dynamics. This framework is formulated as discrete time implementation of the special Kraus representation of completely positive maps. It was already demonstrated with the example of the Toffoli gate and the Quantum Fourier Transform with 3 and 4 qubits that the open quantum walk implementation of the corresponding algorithms outperforms the dissipative quantum computing model (DQC) [8]. In this work we will revise the open quantum walk implementation of a dissipative quantum computing model and with the example of the phase estimation algorithm we will demonstrate the outperformance of the corresponding OQW implementation of the DQC.

The open quantum random walks are formulated as walks on the set of nodes $\mathcal{V}$. The number of nodes is considered to be finite or countable infinite. The Hilbert space of the positions of the walker on the graph is denoted by $\mathcal{K}$ and the set of linearly independent orthonormal vectors $|i\rangle \in \mathcal{K}$ denotes basis. The internal degrees of freedom of the quantum walker, e.g. the spin or n-energy levels, will be described by a separable Hilbert space $\mathcal{H}$ attached to each node of the graph. So that any state of the walker is described by the density operator from the Hilbert space which is a direct product of the Hilbert spaces $\mathcal{H} \otimes \mathcal{K}$.

The transition in the internal degree of freedom of the walker due to the shift from node j

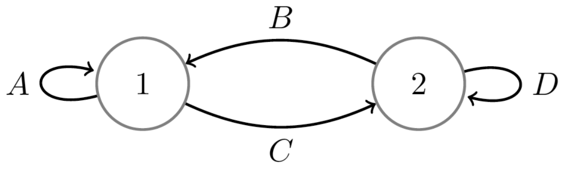

Figure 1. Schematic illustration of a walk on a two-node network, as the simplest non-trivial example of the OQW on the finite graph.

to node i is described by the bounded operator $B_j^i \in \mathcal{H}$. By imposing for each j that,

$$\sum_i {B_j^i}^\dagger B_j^i = I, \tag{1}$$

we make sure, that for each node of the graph $j \in \mathcal{V}$ there is a corresponding completely positive map on the positive operators of $\mathcal{H}$: $\mathcal{M}_j(\tau) = \sum_i B_j^i \tau {B_j^i}^\dagger$. However, the operators B_j^i act only on $\mathcal{H}$ wherefore we can introduce an operator $M_j^i \in \mathcal{H} \otimes \mathcal{K}$, as $M_j^i = B_j^i \otimes |i\rangle\langle j|$. It is shown that, if the condition expressed in Eq. (1) is satisfied, then $\sum_{i,j} {M_j^i}^\dagger M_j^i = 1$. This latter condition defines a CP-map for density matrices on $\mathcal{H} \otimes \mathcal{K}$, i.e.,

$$\mathcal{M}(\rho) = \sum_i \sum_j M_j^i \, \rho \, {M_j^i}^\dagger. \tag{2}$$

The above map defines the discrete time *open quantum walk* (OQW). It is shown [10] that for an arbitrary initial state, the density matrix $\sum_{i,j} \rho_{i,j} \otimes |i\rangle\langle j|$ will take a diagonal form after just one step of the open quantum walk. Hence, we will assume that the initial state of the system has the form $\rho = \sum_i \rho_i \otimes |i\rangle\langle i|$, with $\sum_i \mathrm{Tr}[\rho_i] = 1$. One can easy get an explicit formula for the iteration of the OQW from step n to step $n+1$: $\rho^{[n+1]} = \mathcal{M}(\rho^{[n]}) = \sum_i \rho_i^{[n+1]} \otimes |i\rangle\langle i|$, where $\rho_i^{[n+1]} = \sum_j B_j^i \rho_j^{[n]} {B_j^i}^\dagger$. Generic properties of OQWs have been discussed in [9, 10].

As an illustration of the application of the formalism of OQWs, we consider the walk on a 2-node graph (see Fig. 1). In this case for each node we have:

$$A^\dagger A + C^\dagger C = I, \quad B^\dagger B + D^\dagger D = I. \tag{3}$$

The iteration formula in this case reads,

$$\rho^{[n]} = \rho_1^{[n]} \otimes |1\rangle\langle 1| + \rho_2^{[n]} \otimes |2\rangle\langle 2|, \tag{4}$$

where the particular form of the $\rho_i^{[n]}$ $(i = 1, 2)$ is given by,

$$\begin{aligned} \rho_1^{[n]} &= A\rho_1^{[n-1]}A^\dagger + B\rho_2^{[n-1]}B^\dagger, \\ \rho_2^{[n]} &= D\rho_1^{[n-1]}D^\dagger + C\rho_2^{[n-1]}C^\dagger. \end{aligned} \tag{5}$$

Recently, Verstraete *et al.* [8] proposed a model of a dissipative model of quantum computing, capable of performing universal quantum computation. The dissipative quantum computing setup consists of a linear chain of time registers. Initially, the system is in a time register labeled by 0. The result of the computation is measured in the last time register labeled by T. Neighboring time registers are coupled to local baths. The result of the computation can be read out from the time-register T. In particular, for a quantum circuit given by the set of unitary operators $\{U_t\}_{t=1}^{T}$ the final state of the system is given by $|\psi_T\rangle = U_T U_{T-1} \dots U_2 U_1 |\psi_0\rangle$. It is shown that the unique steady state of the system in this case will be

$$\rho = \frac{1}{T+1} \sum_t |\psi_t\rangle\langle\psi_t| \otimes |t\rangle\langle t|. \tag{6}$$

Recently, it was shown that, using the formalism of OQWs one can perform dissipative quantum computations with higher efficiency. The explicit open quantum walk implementation of the Toffoli gate and the Quantum Fourier Transform has been reported [11]. If one needs to implement a circuit with the set of unitaries so that initial and final states are connected as $|\psi_T\rangle = U_T U_{T-1} \dots U_2 U_1 |\psi_0\rangle$, then the corresponding OQW implementation will be given by a linear chain of $T+1$ nodes with corresponding iteration formula:

$$\rho_j^{[n]} = \omega U_j \rho_{j-1}^{[n-1]} U_j^{\dagger} + \lambda U_{j+1}^{\dagger} \rho_{j+1}^{[n-1]} U_{j+1}, j = 1 \dots (T-1), \tag{7}$$

$$\rho_0^{[n]} = \lambda \rho_0^{[n-1]} + \lambda U_1^{\dagger} \rho_1^{[n-1]} U_1, \tag{8}$$

$$\rho_T^{[n]} = \omega \rho_T^{[n-1]} + \omega U_{T-1} \rho_{T-1}^{[n-1]} U_{T-1}^{\dagger}, \tag{9}$$

where constants ω and λ are positive constants such that $\omega + \lambda = 1$.

In this case the unique steady state of the OQW will have the following form,

$$\rho_{SS} = \sum_{i=0}^{T} p_i |\psi_i\rangle\langle\psi_i|, \tag{10}$$

where p_i denote the steady state probabilities of detecting the walker at the node i. In the case of $\omega = \lambda$ all probabilities of detection will be the same, namely $p_i = 1/(T+1)$. In the case $\omega > \lambda$, the probability of detection of the walker at site T will be bounded between $1/(T+1) < p_T < 1$.

As an illustration of the OQW implementation of the quantum computing model, let us consider the quantum phase estimation algorithm with an unknown unitary operator on a one qubit Hilbert space, and 3 qubits as detection registers [12]. The algorithm consists of three steps: first a Hadamard operation on the measuring qubits (one unitary operation); second the application of conditional shifts with unknown unitary operator (3 unitary operators), and the last step is the inverse Quantum Fourier Transform on the three measuring qubits (9 unitary operations). In total we need to implement 13 unitary operations, this means that the OQW implementation will consist of 14 nodes $(0 \dots 13)$ with readout in the last node 13. The simulation of the OQW implementation of this algorithm is shown in Fig. 2. As in the case of the OQW implementation of the Toffoli gate and the Quantum Fourier Transform [11] the probability of successful read our grows together with the parameter ω.

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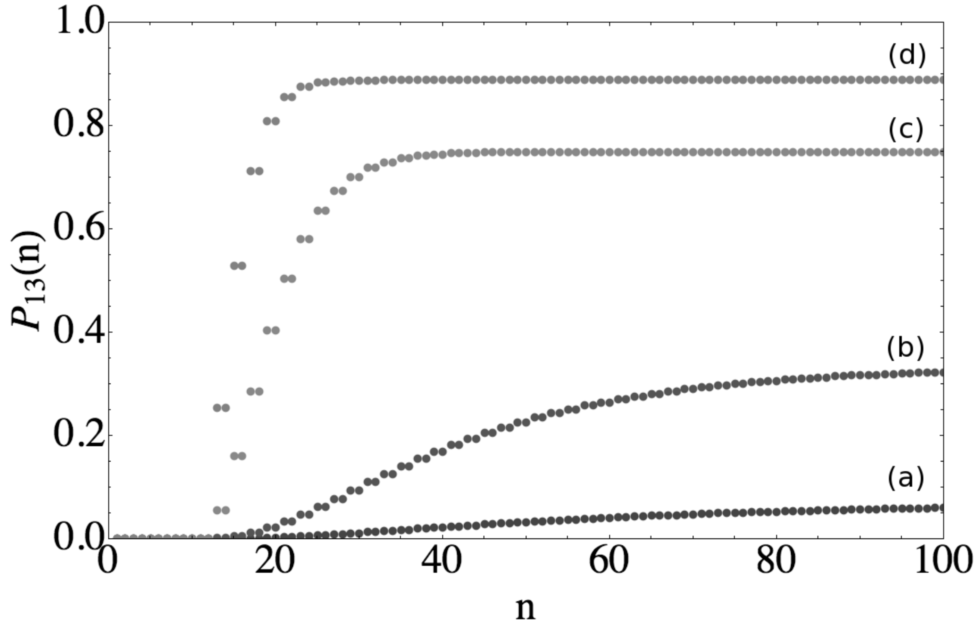

Figure 2. The dynamics of the detection probability in the final node 13 as function of the number of steps of the OQW in the implementation of the quantum phase estimation algorithm. Curves (a) to (d) correspond to different values of the parameter $\omega = 0.5, 0.6, 0.8, 0.9$, respectively.

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