Non-equilibrium steady state entanglement in a continuous variable system

A Ghesquière, I Sinayskiy, F Petruccione

Quantum Research Group, School of Physics, University of KwaZulu-Natal and National Institute for Theoretical Physics, Westville Campus, Durban, 4000, South Africa

E-mail: ghesquiere@ukzn.ac.za, sinayskiy@ukzn.ac.za, petruccione@ukzn.ac.za

Abstract. We examine an entangled bipartite state, where each particle is coupled to an independent reservoir. A difference in temperatures between the baths creates a non-equilibrium steady state. We investigate how the entanglement depends on the temperatures and the strength of the interaction. We find that the steady state does not exist for all values of the system parameters and offer an explanation for this behaviour.

1. Introduction

Dissipation effects induced by the coupling of an entangled quantum system to an environment, may yield various consequences for the entanglement within the system. As such, they have been widely studied in recent years and for a variety of systems. In [15], the authors studied non-equilibrium dissipation effects on a spin chain. Freitas and Paz have studied the dynamics of the Gaussian discord, a measure of quantum correlations, within a system of two oscillators sharing the same heat bath [7]. The effects of detuning the frequencies of two oscillators have also been examined [8].

Our system consists of two identical oscillators, coupled each to a reservoir with different coupling strengths. We keep the reservoirs at different temperatures, which creates a non-equilibrium situation. In many such situations, the steady state can be obtained explicitly and its properties analysed. As such, we examine the steady state behaviour of our bipartite system, particularly, that of its entanglement, so that we might understand the properties of the system in a non-equilibrium situation. Steady state entanglement is found to be difficult to obtain for a system interacting only locally with the environment it is strongly coupled to [12].

To determine the steady state, we use the Non-Rotating-Wave master equation, whose derivation can be found in [2, 9–11, 13]. We measure the entanglement using the logarithmic negativity, which is easy to compute using the covariance matrix formalism described in [1,3,14], which is particularly suited to the study of Gaussian states.

The stationary state is determined in Section 2. We will examine some results and offer some concluding remarks in Section 3.

2. Steady State

We study an entangled pair of identical oscillators, each one oscillating with frequency ω_0 and mass m, and coupled to its own heat bath. They have positions and momenta x_1, x_2, p_1 and

p_2. The oscillators are interacting linearly with each other with coupling strength κ. The total Hamiltonian reads as

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{m\omega_0^2}{2}(x_1^2 + x_2^2) + \kappa x_1 x_2 + \sum_j \left\{ \frac{p_j^2}{2m_j} + \frac{m_j\omega_j^2}{2}(q_j - x_1)^2 \right\} + \sum_k \left\{ \frac{p_k^2}{2m_k} + \frac{m_k\omega_k^2}{2}(q_k - x_2)^2 \right\}. \quad (1)$$

$q_{j,k}$ and $p_{j,k}$ are the baths' positions and momenta; ω's are the baths' oscillators frequencies and m_j and m_k are their masses. The interaction between the reservoirs and the particles is considered to be linear in their position; the Non-Rotating-Wave master equation for the system's density matrix ρ, in the quantum Brownian limit is written as [2, 9–11, 13]

$$\dot{\rho} = -\frac{i}{\hbar}\left[H_s, \rho\right] - \frac{i\gamma_1}{2\hbar}\left[x_1, [p_1, \rho]_+\right] - \frac{\gamma_1 k T_1}{\hbar^2}\left[x_1, [x_1, \rho]\right] - \frac{i\gamma_2}{2\hbar}\left[x_2, [p_2, \rho]_+\right] - \frac{\gamma_2 k T_2}{\hbar^2}\left[x_2, [x_2, \rho]\right], \quad (2)$$

where the T_i $(i = 1, 2)$ are the temperatures of the reservoirs, and the γ_i $(i = 1, 2)$ the coupling constants between the particles and their respective reservoirs. To study the entanglement, we will use the logarithmic negativity which may be expressed in terms of the covariance matrix as follows [16]

$$\mathcal{L}_{\mathcal{N}}(\rho) = -\sum_{i=1}^{n} \log_2(\min(1, |\lambda_i^{T_1}|))$$

where the λ's here are the symplectic eigenvalues of the covariance matrix, whose terms may be determined as $\Gamma_{jk} = 2\mathrm{Re}\,\mathrm{Tr}\left[\rho(\hat{R}_j - \langle\hat{R}_j\rangle)(\hat{R}_k - \langle\hat{R}_k\rangle)\right]$. The $\hat{R}$'s are the system's operators, e.g. $\hat{x}$ and $\hat{p}$. We note here that since we study a Gaussian state, we may determine the covariance matrix in terms of second moments only as $\Gamma_{jk} = 2\mathrm{Re}\,\mathrm{Tr}\left[\rho\hat{R}_j\hat{R}_k\right]$. We write a system of equations for the terms of the covariance matrix which yields the solution Γ_{ss} in the stationary limit

$$\begin{aligned}
\langle x_1^2\rangle &= \frac{-\gamma_1 k T_1(\gamma_2^2 m^2\omega_0^4 - \gamma_2^2\kappa^2 + \gamma_1\gamma_2 m^2\omega_0^4 + \kappa^2\omega_0^2) - \gamma_2 k T_2 \kappa^2(\omega_0^2 + \gamma_1\gamma_2)}{(\kappa^2 - m^2\omega_0^4)(\gamma_1 + \gamma_2)(\gamma_1\gamma_2 m^2\omega_0^2 + \kappa^2)}, \\
\langle x_2^2\rangle &= \frac{-\gamma_1 k T_1 \kappa^2(\omega_0^2 + \gamma_1\gamma_2) - \gamma_2 k T_2(\gamma_1^2 m^2\omega_0^4 - \gamma_1^2\kappa^2 + \gamma_1\gamma_2 m^2\omega_0^4 + \kappa^2\omega_0^2)}{(\kappa^2 - m^2\omega_0^4)(\gamma_1 + \gamma_2)(\gamma_1\gamma_2 m^2\omega_0^2 + \kappa^2)}, \\
\langle p_1^2\rangle &= \frac{\gamma_1 k T_1(\gamma_2^2 m^2\omega_0^2 + \gamma_1\gamma_2 m^2\omega_0^2 + \kappa^2) + \gamma_2 k T_2 \kappa^2}{(\gamma_1 + \gamma_2)(\gamma_1\gamma_2 m^2\omega_0^2 + \kappa^2)}, \\
\langle p_2^2\rangle &= \frac{\gamma_1 k T_1 \kappa^2 + \gamma_2 k T_2(\gamma_1^2 m^2\omega_0^2 + \gamma_1\gamma_2 m^2\omega_0^2 + \kappa^2)}{(\gamma_1 + \gamma_2)(\gamma_1\gamma_2 m^2\omega_0^2 + \kappa^2)}, \\
\langle x_1 x_2\rangle &= \frac{-\gamma_1 k T_1 \kappa - \gamma_2 k T_2 \kappa}{m(\gamma_1 + \gamma_2)(\kappa^2 - m^2\omega_0^4)}, \\
\langle x_1 p_2\rangle &= \langle p_2 x_1\rangle = -\langle x_2 p_1\rangle = -\langle p_1 x_2\rangle = \frac{\gamma_1 k T_1 \gamma_2 \kappa - \gamma_2 k T_2 \gamma_1 \kappa}{(\gamma_1 + \gamma_2)(\gamma_1\gamma_2 m^2\omega_0^2 + \kappa^2)},
\end{aligned} \quad (3)$$

and $\langle[x_1, p_1]_+\rangle = \langle[x_2, p_2]_+\rangle = \langle p_1 p_2\rangle = 0$. Using $[x, p] = i\hbar$, it is clear that $\mathrm{Re}[\langle x_i p_i\rangle] = \mathrm{Re}[\langle p_i x_i\rangle] = \frac{1}{2}\mathrm{Re}[\langle [x_i, p_i]_+\rangle] = 0$ $(i = 1, 2)$. The steady state covariance matrix can now be witten as

$$\Gamma_{ss} = \begin{pmatrix} 2\langle x_1^2\rangle & 0 & 2\langle x_1x_2\rangle & \langle x_1p_2\rangle \\ 0 & 2\langle p_1^2\rangle & -\langle p_1x_2\rangle & 0 \\ 2\langle x_2x_1\rangle & -\langle x_2p_1\rangle & 2\langle x_2^2\rangle & 0 \\ \langle p_2x_1\rangle & 0 & 0 & 2\langle p_2^2\rangle \end{pmatrix} . \tag{4}$$

One can easily notice from (3) that there is a singularity at $\kappa = \pm m\omega_0^2$. The Quantum Langevin Equation [6] helps us to understand this. For simplicity, let us set $\gamma_1 = \gamma_2 = \gamma$ and apply the normal mode transformation $X_\pm = (x_1 \pm x_2)/\sqrt{2}$ so that in the high temperature memoryless case, that we consider, the Quantum Langevin Equation has the form

$$\begin{aligned} m\ddot{X}_+ + \gamma\dot{X}_+ + (m\omega_0^2 + \kappa)X_+ &= \Xi_+(t)\,, \\ m\ddot{X}_- + \gamma\dot{X}_- + (m\omega_0^2 - \kappa)X_- &= \Xi_-(t)\,, \end{aligned} \tag{5}$$

where $\Xi_\pm(t) = (\xi_1(t) \pm \xi_2(t))/\sqrt{2}$ and ξ_j are the random forces between the particles and their respective baths, with autocorrelation function given by

$$\frac{1}{2}\langle\big[\xi(t), \xi(t')\big]_+\rangle = 2kT\gamma\delta(t-t')\,.$$

If $\kappa = \pm m\omega_0^2$, then one of the oscillators becomes effectively an unbounded classical Brownian particle. Such system will not reach a steady state.

3. Observations and concluding remarks

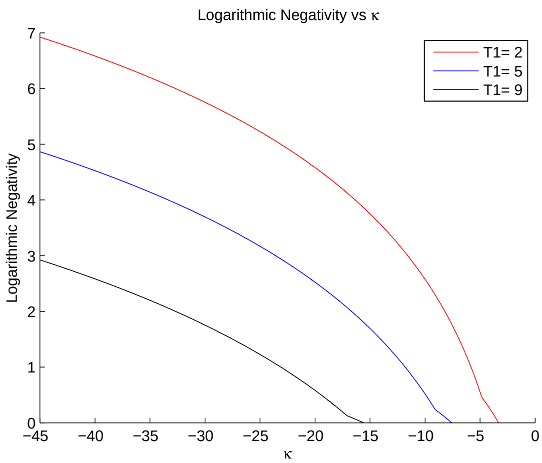

Figure 1: Logarithmic negativity versus κ for $T_2 = 5$, $\gamma_1 = 0.1$, $\gamma_2 = 0.1$, $m = 3$ and $\omega_0 = 0.5$.

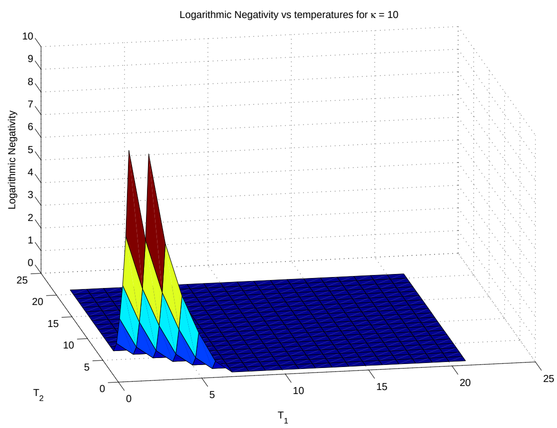

Figure 2: Logarithmic negativity versus the temperatures for $\kappa = 10$, $\gamma_1 = 0.1$, $\gamma_2 = 0.1$, $m = 3$ and $\omega_0 = 0.5$.

We observe on Figure 1, how the entanglement behaves, when plotted against κ for various temperatures. It is easily noticed that as κ approaches zero, the entanglement decreases. Furthermore, the entanglement is less as the temperature is higher. We observe on Figure 2 the effect of the temperatures on the entanglement for a given κ. It can be seen that as the temperatures increase, the entanglement vanishes. However, one may also notice on Figures 3 and 4 that the entanglement disappears for higher temperatures if κ is larger. This suggests

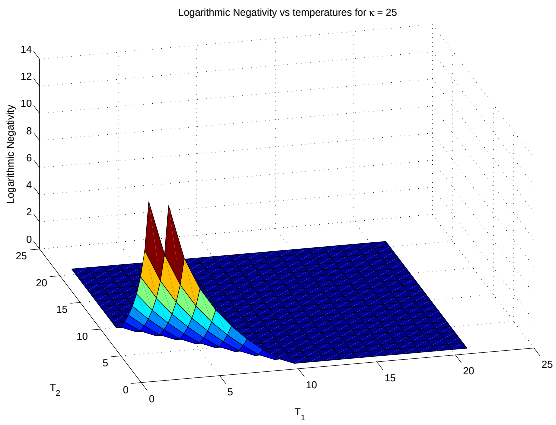

Figure 3: Logarithmic negativity versus the temperatures for $\kappa = 25$, $\gamma_1 = 0.1$, $\gamma_2 = 0.1$, $m = 3$ and $\omega_0 = 0.5$.

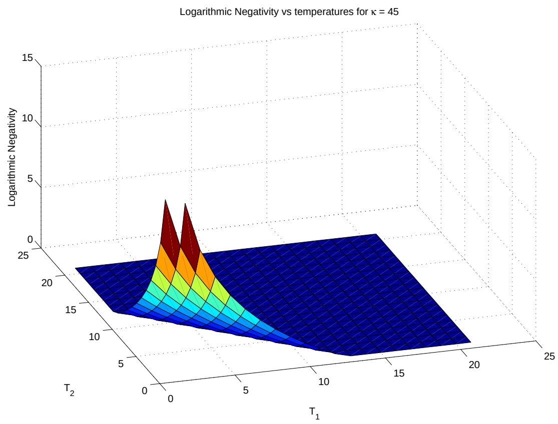

Figure 4: Logarithmic negativity versus the temperatures for $\kappa = 45$, $\gamma_1 = 0.1$, $\gamma_2 = 0.1$, $m = 3$ and $\omega_0 = 0.5$.

that a much stronger interaction is needed to counteract the effects of the temperatures. On the other hand, at $\kappa = 0$, there is no entanglement. Similar observations have been reported in [4, 5] and [11]. It may be worth noting that the steady state entanglement is that which is created through interaction between the system and their reservoirs. Indeed, it is also known as thermal entanglement and is independent of any initial entanglement the state may have been initially prepared with.

We have determined explicitly the steady state of a system of two oscillators, coupled to two independent reservoirs. However, we have shown that this solution does not exist for two particular values of system parameters. We have provided a simple explanation of the phenomenon. We have also shown that if the baths are at high temperatures, then the interaction between the particles must be strong in order for there to be steady state entanglement.

Acknowledgements

This work is based on research supported by the South African Research Chair Initiative of the Department of Science and Technology and National Research Foundation.

References

[1] J. Anders. *LANL e-print quant-ph/0610263*, 2003.
[2] H. P. Breuer and F. Petruccione. *The Theory of Open Quantum Systems*. Oxford: Oxford Press, 2002.
[3] J. Eisert and M. B. Plenio. *International Journal of Quantum Information*, 1:479, 2003.
[4] Z. Ficek and R. Tanás. *Physical Review A*, 74:024304, 2006.
[5] Z. Ficek and R. Tanás. *Physical Review A*, 77:054301, 2008.
[6] G. W. Ford, J. T. Lewis, and R. F. O'Connell. *Physical Review A*, 37:11, 1988.
[7] J. N. Freitas and J. P. Paz. *Physical Review A*, 85:032118, 2012.
[8] F. Galve, G. L. Giorgi, and R. Zambrini. *Physical Review A*, 81:062117, 2010.
[9] C. W. Gardiner and P. Zoller. *Quantum Noise*. Springer, 2nd enlarged edition edition, 2000.
[10] A. Ghesquière. *Entanglement in a Bipartite Gaussian State*. PhD thesis, available as LANL e-print quant-ph/1105.0564, 2009.
[11] A. Ghesquière and T. Dorlas. Entanglement of a two-particle gaussian state interacting with a heat bath. *Physics Letters A*, 377:1682, 2013.
[12] L. Hartmann, W. Dür, and H.-J. Briegel. *Physical Review A*, 74:052304, 2006.
[13] W. J. Munro and C. W. Gardiner. *Physical Review A*, 53:4, 1996.
[14] M. B. Plenio, J. Hartley, and J. Eisert. *New Journal of Physics*, 6:36, 2004.

[15] I. Sinaysky, D. Burgarth, and F. Petruccione. *Physical Review A*, 78:062301, 2008.
[16] G. Vidal and R. F. Werner. *Physical Review A*, 65:3, 2002.