

Computer assisted ‘proof’ of the global existence of periodic orbits in the Rössler system

André E Botha¹ and Wynand Dednam^{1,2}

¹Department of Physics, University of South Africa, Science Campus, Private Bag X6, Florida Park 1710, South Africa

²Departamento de Física Aplicada Universidad de Alicante, San Vicente del Raspeig, E-03690 Alicante, Spain

E-mail: bothaae@unisa.ac.za

Abstract. The numerical optimized shooting method for finding periodic orbits in nonlinear dynamical systems was employed to determine the existence of periodic orbits in the well-known Rössler system. By optimizing the period T and the three system parameters, a , b and c , simultaneously, it was found that, for any initial condition $(x_0, y_0, z_0) \in \mathbb{R}^3$, there exists at least one set of optimized parameters corresponding to a periodic orbit passing through (x_0, y_0, z_0) . After a discussion of this result it was concluded that its analytical proof may present an interesting new mathematical challenge.

1. Introduction

The three-dimensional Rössler system (RS) was originally conceived as a simple prototype for studying chaos [1]. With only one nonlinear term it can be thought of as a simplification of the well-known Lorenz system [2] and as a minimal model for continuous-time chaos [3].

Over the past 38 years the RS has been studied extensively, both in its own right [4–15] and as a model, to illustrate various nonlinear phenomena and different types of chaos [16–19]. Surprisingly, despite the literally thousands of scientific articles that have been devoted to the RS, it still poses several open questions. For example, although the conjecture by Leonov [20], about the Lyapunov dimension of the RS (and two other types of Rössler systems), has recently been verified numerically [21], the exact Lyapunov dimension for the RS is not known analytically. Leonov [22] has recently also provided an analytical estimate of the attractor dimension in the RS, however, similar estimates for slightly more general Rössler systems remain a difficult open problem. There is also an ongoing discussion about whether or not the RS can indeed produce a strange attractor, in the same sense as that of the Lorenz system [17, 23].

More pertinent to the present study are the recent articles on the existence or non-existence and/or classification of the periodic orbits of the RS, in terms of its three control parameters a , b and c . Starkov and Starkov [24] have shown that, when the parameters satisfy the condition $c^2 - 4ab < 0$, no periodic solutions can exist. They have also proved that, when $ac < 0$ ($ac > 0$) and $ab > 0$, the periodic orbits exist entirely in the negative (positive) half space, i.e. $z < 0$ ($z > 0$).

There have been several attempts to demonstrate the existence of orbits more generally; such as those by Genesisio and Ghilardi [25], in which the existence of the quasi-periodic orbits in

third-order systems like those of Rössler, Lorenz and Chua *et al.* [26] are proved. More recently, instead of the focus of such studies being on the unstable orbits, the experimentally observable chaotic and stable periodic orbits of simple chaotic flows, including the RS, have been classified very generally in terms of their control parameters [16]. In the parameter space, families of stable periodic solutions have been found to self-organize about the so-called isolated periodicity hubs of the RS. (See Ref. [16] and the references therein.)

Another active area of related research is on the integrability of the RS [10–12, 14, 27, 28]. In Ref. [12] it was shown that Darboux integrability of the RS for various parameter values leads to surfaces in state space containing periodic orbits. The bifurcations and routes to chaos in the RS have also been investigated [9, 29] and there is at least one exhaustive study on the complete parametric evolution of the system [6]. (Also see Ref. [5] and the references therein.)

The present work developed as a result of using the RS as a test case for our recently-developed numerical technique (called the optimized shooting method) for finding periodic solutions in nonlinear dynamical systems [30]. During the course of testing our method we discovered that it could optimize the system parameters to find at least one periodic orbit for *any* initial condition. At first this finding was very surprising to us, and we assumed that it was most likely due to a coding error. We therefore tested the codes more thoroughly and also experimented with a variety of different integration schemes and platforms, in order to verify the results independently. After confirming that our numerical results were indeed accurate, we turned to the literature to establish if a similar result had been reported elsewhere for the RS. However, in all of the related literature we found, there appeared to be no explicit proof (or mention) of the result we had obtained. Hence we consider it worthy of a separate brief report.

We are certainly not the first to use numerical computations to ‘prove’, i.e. motivate, the existence of certain analytical properties. Pilarczyk [18], for example, provides a computer assisted ‘proof’ of the existence of a periodic orbit in the RS. Wilczak and Zgliczyński [19] have also made use of a numerical method to ‘prove’ the existence of two period-doubling bifurcations in Rössler’s system, as well as the existence of a branch of period two points connecting them.

The remaining material in this article is organized as follows. In Sec. 2 we provide a brief description of the optimized shooting method. In Sec. 3 we state the main result in the form of a conjecture and describe the procedure that was followed in order to motivate it. Section 4 ends with a short discussion and conclusion.

2. Optimized shooting method

Today there are a variety of numerical methods available for finding the periodic orbits of nonlinear dynamical systems [31]. For the present purpose we make use of the optimized shooting method [30], which originally enabled us to discover the reported new property of the periodic orbits in the RS [1]. Briefly, this method makes use of Levenberg-Marquardt optimization to find the periodic orbits by minimizing a residual (or error function) [32, 33].

To apply the method to the RS, we re-write the system equations as

$$\dot{x} = T(-y - z), \quad \dot{y} = T(x + ay), \quad \dot{z} = T(b + z(x - c)), \quad (1)$$

where T is the (as yet) unknown period of the desired solution and the overdot indicates differentiation with respect to the dimensionless time $\tau = t/T$. Since one period corresponds to integration over τ from zero to one, we define the residual as

$$\mathbf{R} = (\mathbf{x}(1) - \mathbf{x}(0), \mathbf{x}(1 + \Delta\tau) - \mathbf{x}(\Delta\tau), \dots, \mathbf{x}(1 + p\Delta\tau) - \mathbf{x}(p\Delta\tau)), \quad (2)$$

where $\Delta\tau$ is a fixed integration step size, $p = 0, 1, 2, \dots$ and $\mathbf{x} \equiv (x, y, z)$. The residual is a function of the initial conditions, period, and three system parameters; since it depends on these through the solution to (1). Furthermore, since $\mathbf{x}(\tau) = \mathbf{x}(\tau + 1)$ for periodic solutions, it

can immediately be seen that such solutions correspond to a vanishing residual, i.e. the periodic solutions are found by optimizing \mathbf{R} to be as close as possible to zero.

3. Results

We state the main result of this work in the form of a conjecture.

Conjecture: For any initial condition $(x_0, y_0, z_0) \in \mathbb{R}^3$, there exists real parameters, a , b and c , for (1), such that its solution is periodic.

For initial conditions on plane $z = 0$, inspection of (1) shows that the conjecture is trivially satisfied: for $a = b = c = 0$ the solutions are circles, with periods equal to 2π . The analytical proof of the integrability of the system, for this case, is given in Refs. [15] and [11].

Off the $z = 0$ plane the validity of the conjecture is established numerically by performing optimization of the four parameters, T , a , b and c , for sets of randomly selected initial conditions. To facilitate the selection of these initial conditions it is convenient to rewrite (1) in terms of scaled (primed) coordinates, defined by $x = \alpha x'$, $y = \alpha y'$, and $z = \alpha z'$. Here $\alpha \geq 1$ is the scale factor. In terms of the primed coordinates (1) is given by

$$\dot{x}' = T(-y' - z'), \quad \dot{y}' = T(x' + ay'), \quad \dot{z}' = T(b/\alpha + \alpha z'(x' - c/\alpha)). \quad (3)$$

For a given α , the procedure for selecting the initial conditions then consisted of generating 100 distinct random points, within or on the unit sphere. For each initial condition the optimized shooting method was used to find the four parameters that produce a periodic orbit. Starting from $\alpha = 1$, the value of the scale factor was increased, in steps of 1, up to the maximum value of 100. This procedure produced a total of 100×100 random initial conditions. All 10^4 initial conditions were successfully optimized, i.e. in each case the magnitude of \mathbf{R} was successfully optimized to be below the set numerical tolerance ($|\mathbf{R}| < 10^{-12}$ in this calculation).

Table 1 lists the optimized parameters for 13 arbitrarily chosen periodic orbits. For ease

Table 1. Thirteen of the 10^4 periodic orbits found for the Rössler system via the optimized shooting method. The column headings are as follows: n is the orbit number (used to refer to the orbit in the main text), α is the scale parameter introduced in (3), x'_0 , y'_0 and z'_0 are the primed coordinates of the randomly selected initial conditions, T is the period, and the last three columns give the optimized system parameters.

n	α	x'_0	y'_0	z'_0	T	a	b	c
1	1	-0.322	0.283	-0.827	6.285	-0.01832876699	-37.861974659	45.448602039
2	1	0.083	0.498	0.756	6.282	0.01659280123	34.3119795334	45.496814976
3	1	0.271	-0.033	-0.492	6.284	-0.01074867143	-22.256338664	45.495734090
4	1	-0.843	-0.154	-0.338	6.283	-0.00756927443	-15.661740755	45.483293583
5	1	0.920	-0.101	-0.062	6.285	-0.00133598606	-2.7619826425	45.463987246
6	10	0.274	-0.843	-0.019	6.280	-0.00346652925	-9.6387456311	53.299901841
7	10	-0.431	0.429	-0.132	6.287	-0.02714795455	-75.519091787	52.954713982
8	10	-0.762	0.094	-0.023	6.283	-0.00504377136	-13.929997631	52.959008822
9	10	0.721	0.212	0.532	6.303	0.09058831072	242.356061832	52.926271854
10	10	0.212	-0.008	-0.946	6.412	-0.18855060455	-483.78516338	53.074125658
11	100	0.370	0.525	-0.489	9.221	-0.72151348243	-2913.5197570	96.641076651
12	100	0.465	-0.177	0.111	6.506	0.27456932042	1.25094211568	35.351207642
13	100	0.045	-0.881	0.206	1.670	0.28422944418	0.06378881294	25.148204118

of presentation, the randomly chosen initial conditions were first rounded off to three decimal places, before applying the optimized shooting method. Thus the initial conditions, as listed in columns 3-5 of table 1, are exact. On the other hand the periods listed in column 6 are given approximately to three decimal places. Since they are not required to reconstruct the orbits, their full 11 digit accuracy is not required here.

Two other features of table 1 are also worth mentioning. First, we notice that the signs of the parameters a and b are always the same as that of z'_0 . This is a property of all the found orbits and it confirms the analytical result proved by Starkov and Starkov [24]: when $ac < 0$ ($ac > 0$) and $ab > 0$, the periodic orbits exist entirely in the negative (positive) half space. Second, in Ref. [24] it was also shown that for $c^2 - 4ab < 0$, no periodic orbits can exist. In table 1, and indeed for all the found orbits, $c^2 - 4ab$ is positive. Thus our results are consistent with the known classifications for periodic orbits in the RS.

In total more than 100 spot checks were made on randomly selected orbits taken from the set of 10^4 . Such checks were made by plotting and visually inspecting the selected orbits. Figure 1 shows the plots for four orbits that are listed in table 1, i.e. orbits numbers 1, 9, 12 and 13.

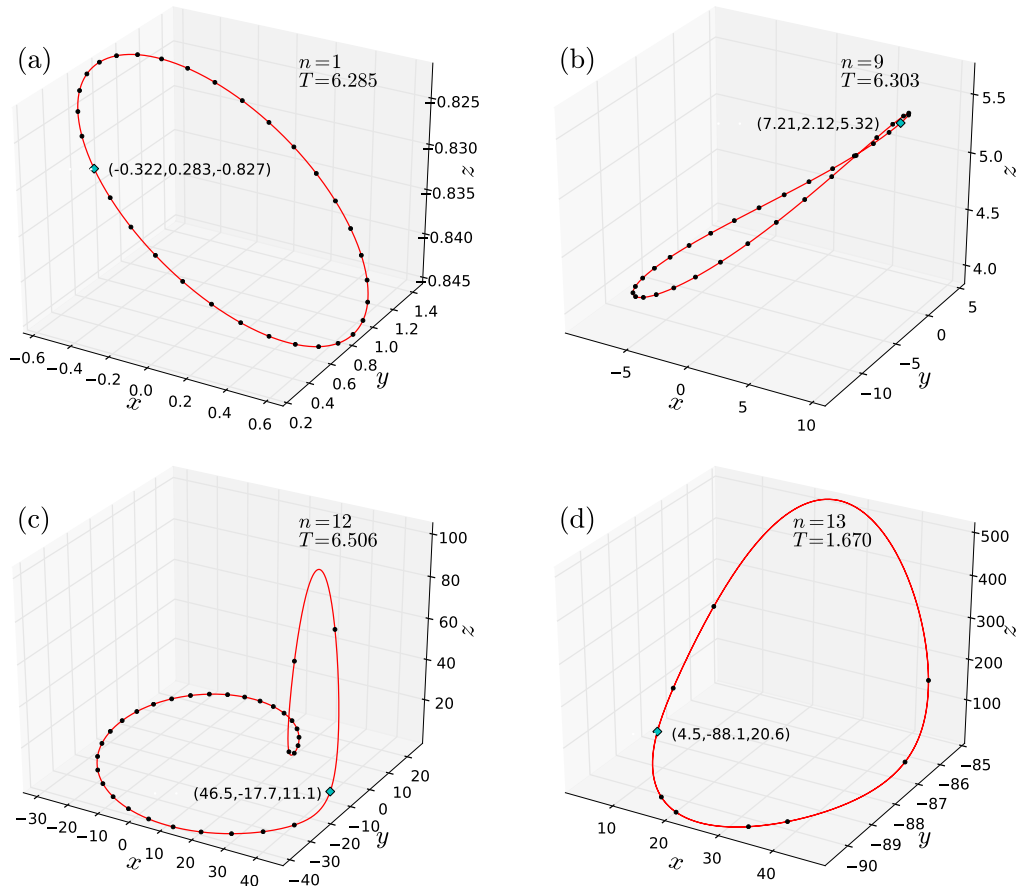


Figure 1. Phase portraits of four arbitrarily selected periodic orbits, chosen from table 1. The initial condition for each orbit is indicated by a blue square marker. Black filled circular markers indicate points along the orbit, every 200 integration time steps. Since a fixed time step of $\Delta\tau = 1/1024$ was used, the circular markers provide an indication of the velocity $(\dot{x}(\tau), \dot{y}(\tau), \dot{z}(\tau))$ along each orbit.

In figure 1(a) it can be seen that orbit one is confined to lie within the negative half space

$\mathbb{R}^3 - \{z \geq 0\}$, while orbits 9, 12 and 13, shown in figures (b)-(d), occur in the positive half space $\mathbb{R}^3 - \{z \leq 0\}$. This observation is consistent with the form of (1). If periodic orbits existed that crossed from one half space into the other, they would have to intersect the plane $z = 0$ more than once. Moreover, at $z = 0$ the sign of \dot{z} would have to be positive (negative) in going from the negative (positive) half space to the positive (negative) half space. However, the form of (1) clearly excludes this possibility, since for $z = 0$ the sign of \dot{z} is determined by the parameter b . Therefore, for a given set of parameters, the sign of \dot{z} at $z = 0$ cannot change. Hence, the non-trivial periodic orbits are confined to lie entirely within one half space or the other.

Finally, as an alternative way (i.e. without using the residual) of estimating the accuracy to which the optimized shooting method is capable of determining the periodic solutions, several initial conditions with $z = 0$ were also examined. In all such cases the magnitudes of the largest optimized system parameters were found to be less than 10^{-11} , with the expected period of 2π also given accurately to eleven decimal places.

4. Discussion and conclusion

We have established a computer assisted ‘proof’ of the global existence of periodic solutions in the Rössler system. By global we mean that, for any initial condition $(x_0, y_0, z_0) \in \mathbb{R}^3$, our numerical method was able to optimize simultaneously the period T and system parameters, a , b and c , in order to find at least one periodic solution passing through the initial condition. We have stated this result as a conjecture, as opposed to a proposition, because the latter term is generally reserved for mathematical results that can be proved rigorously. The term ‘proof’, which we have written consistently in single quotes, should thus be understood to mean a numerical justification/motivation, rather than a mathematical proof. This is also the sense in which the same term is used elsewhere in the literature on computer assisted proofs. (See, for example, Refs. [18] and [19].)

It is interesting to note that, in some cases, more than one solution passing through a given initial condition could be found, i.e. two or more distinct sets of the optimized parameters could be found for a given initial condition. For example, in the case of orbit number 12 (see figure 1(c) and table 1), there also exists a different period orbit passing through the same initial point, with precisely twice the period, i.e. $T = 13.0116560813$, at slightly different system parameters. This multiplicity, of period doubled orbits all passing through the same point, offers an interesting new possibility. Whereas conventionally the universal period doubling route to chaos (the so-called Feigenbaum scenario [34, 35]) is usually achieved by varying one system parameter at a time, we see here that it may also be possible (by following a very specific (three dimensional) path in the parameter space) to obtain a special sequence of period doubling orbits *which all pass through the same point in the phase space*. It remains to be seen whether such a special sequence would also lead to chaos and follow Feigenbaum’s universal scaling laws.

In conclusion, we have formulated a conjecture about the global existence of periodic solutions in Rössler’s system. Our conjecture is numerically supported (i.e. ‘proved’) and it postulates the existence of periodic orbits passing through *any* point in the phase space, for a suitable choice of the system parameters. While it is in principle possible to establish this result analytically, to date there does not appear to be any mention in the literature of either the result itself or its proof. Unfortunately we were not able to construct such an analytical proof ourselves. However, in view of the compelling computational evidence which we have provided, we hope that some theoretically inclined readers may find it interesting to devise a rigorous mathematical proof of our conjecture.

Acknowledgments

The financial assistance of the National Research Foundation of South Africa towards this research is hereby acknowledged.

References

- [1] Rössler O E 1976 An equation for continuous chaos *Phys. Lett. A* **57** 397–8
- [2] Lorenz E N 1963 Deterministic nonperiodic flow *J. Atmos. Sci.* **20** 130–41
- [3] Gaspard P 2005 Rössler systems *Encyc. Nonlin. Sci.* ed Scott A (New York: Routledge) pp 808–11
- [4] Algaba A *et al.* 2007 Resonances of periodic orbits in Rössler system in presence of a triple-zero bifurcation *Int. J. Bifurcation Chaos* **17** 1997–2008
- [5] Barrio R, Blesa F, Dena A and Serrano S 2011 Qualitative and numerical analysis of the Rössler model: bifurcations of equilibria *Computers and Mathematics with Applications* **62** 4140–50
- [6] Barrio R, Blesa F and Serrano S 2009 Qualitative analysis of the Rössler equations: bifurcations of limit cycles and chaotic attractors *Physica D* **238** 1087–1100
- [7] Castro V *et al.* 2007 Characterization of the Rössler system in parameter space *Int. J. Bifurcation Chaos* **17** 965–73
- [8] Galias Z 2006 Counting low-period cycles for flows *Int. J. Bifurcation Chaos* **16** 2873–86
- [9] Gardini L 1985 Hopf bifurcations and period-doubling transitions in Rössler model *Il Nuovo Cimento B* **89** 139–60
- [10] Lima M F S and Llibre J 2011 Global dynamics of the Rössler system with conserved quantities *J. Phys. A: Math. Theor.* **44** 365201 (15pp)
- [11] Lliubre J and Valls C 2007 Formal and analytic integrability of the Rössler system *Int. J. Bifurcation Chaos* **17** 3289–93
- [12] Lliubre J and Zhang X 2002 Darboux integrability for the Rössler system *Int. J. Bifurcation Chaos* **12** 421–28
- [13] Teryokhin M T and Panfilova T L 1999 Periodic solutions of the Rössler system *Russian Mathematics* **43** 66–9
- [14] Tudoran R M and Girban A 2012 On the completely integrable case of the Rössler system *J. Math. Phys.* **53** 052701 (10pp)
- [15] Zhang X 2004 Exponential factors and Darboux integrability for the Rössler system *Int. J. Bifurcation Chaos* **14** 4275–83
- [16] Gallas J A C 2010 The structure of infinite periodic and chaotic hub cascades in phase diagrams of simple autonomous flows *Int. J. Bifurcation Chaos* **20** 197–211
- [17] Lozi R 2013 *Topology and Dynamics of Chaos* ed Letellier C and Gilmore R (Singapore: World Scientific)
- [18] Pilarczyk P 1999 Computer assisted method for proving existence of periodic orbits *Topological Methods in Nonlinear Analysis* **13** 365–77
- [19] Wilczak D and Zgliczyński P 2009 Period doubling in the Rössler system – a computer assisted proof *Found. Comput. Math.* **9** 611–49
- [20] Leonov G A 2012 Lyapunov functions in the attractors dimension theory *J. Appl. Math. Mech.* **76** 129–41
- [21] Kuznetsov N V, Mokaev T N and Vasilyev P A 2014 Numerical justification of Leonov conjecture on Lyapunov dimension of Rössler attractor *Commun. Nonlinear Sci. Numer. Simulat.* **19** 1027–34
- [22] Leonov G A 2014 Rössler systems: estimates for the dimension of attractors and homoclinic orbits *Doklady Mathematics* **89** 1–3
- [23] Zgliczyński P 1997 Computer assisted proof of chaos in the Rössler equations and in the Hénon map *Nonlinearity* **10** 243–52

- [24] Starkov K E and Starkov K K 2007 Localization of periodic orbits of the Rössler system under variation of its parameters *Chaos, Solitons and Fractals* **33** 1445–49
- [25] Genesio R and Ghilardi C 2005 On the onset of quasi-periodic solutions in third-order nonlinear dynamical systems *Int. J. Bifurcation Chaos* **15** 3165–80
- [26] Chua L, Komuro M and Matsumoto T 1986 The double scroll family *IEEE Trans. Circuits Syst.* **33** 1073–118
- [27] Chandrasekar V, Senthilvelan M and Lakshmanan M 2009 On the complete integrability and linearization of nonlinear ordinary differential equations. III. Coupled first-order equations *Proc. R. Soc. A* **465** 585–608
- [28] Hitchin N 1997 *Integrable systems: An Introduction* (Oxford: Notes from the Mathematical Institute)
- [29] Nikolov S and Petrov V 2004 New results about route to chaos in Rössler system *Int. J. Bifurcation Chaos* **14** 293–308
- [30] Dednam W and Botha A E 2014 Optimized shooting method for finding periodic orbits of nonlinear dynamical systems *Engineering with Computers* DOI: 10.1007/s00366-014-0386-6
- [31] Abad A, Barrio R and Dena A 2011 Computing periodic orbits with arbitrary precision *Phys. Rev. E* **84** 016701 (6pp)
- [32] Levenberg K 1944 A method for the solution of certain problems in least squares *Quart. Appl. Math.* **2** 164–68
- [33] Marquardt D W 1963 An algorithm for least-squares estimation of nonlinear parameters *SIAM J. Appl. Math.* **11** 431–41
- [34] Feigenbaum M J 1978 Quantitative universality for a class of nonlinear transformations *Journal of Statistical Physics* **19** 25–52
- [35] Feigenbaum M J 1979 The universal metric properties of nonlinear transformations *Journal of Statistical Physics* **21** 669–706