Wilson lines and color-neutral operators in the color glass condensate

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Abstract. Color-neutral operators (singlets) are useful in high energy QCD to construct Wilson line correlators. In this proceedings we describe an algorithm that allows us to systematically construct all singlets for a given Fock-space configuration. In particular, we exemplify this algorithm with the singlets of the $3q + 3\bar{q}$-algebra.

1. Introduction

Confinement is a phenomenon of QCD, that is to this day theoretically not well understood in the sense that it is unclear, what feature of QCD forces color-charged objects to be confined. However, the consequences of confinement are very clear: all objects that carry color charge (i.e. quarks and gluons) are necessarily bound together in color-neutral states such as baryons or mesons. It is thus of paramount importance to understand color-neutral states, so-called *singlet-states* in order to perform QCD calculations on physically meaningful objects.

The study of singlets becomes even more relevant when considered in the high-energy limit of QCD. In this regime, all interactions between a projectile and a target occur (to a good approximation) exclusively via Wilson lines. This is due to the fact that the target is highly Lorentz contracted and thus acts as an essentially static but extremely localized color source, allowing us to make a no-recoil approximation, [1].

We begin our discussion with an explanation of why the study of singlets is so significant in this branch of physics. In many places, we will do so by summarizing results of [1]; we have used the same diagrammatic notation as [1] in order to make a comparison easier. Beyond this section, the diagrammatic notation will be modified to better suit our purposes.

Wilson line correlators can carry singlets into singlets. This is easiest seen when considering the dipole correlator; this example is described in detail in [1], we will therefore only give a brief summary here. The contribution of the dipole interaction with a target via a Wilson line to the total cross section is described in the absolute value of the square of the difference between the state in which an interaction takes place, and the state which describes no interaction with the

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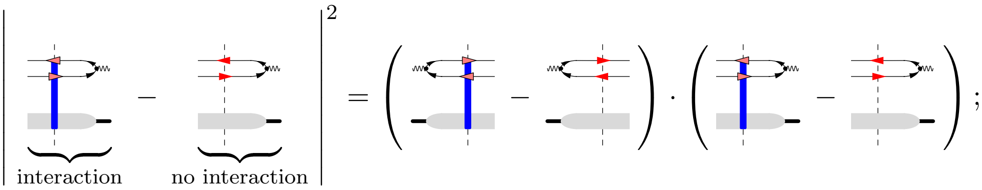
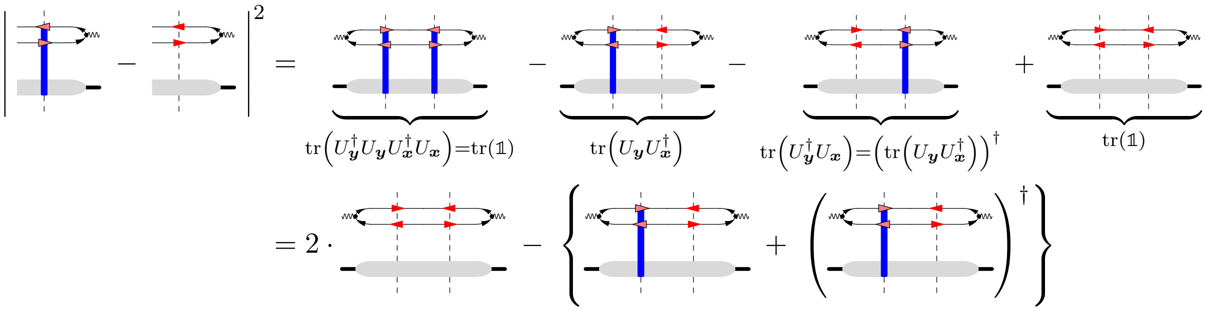
target,



(1)

in the above diagrams, the blue line represents an interaction with the target, while the dashed line indicates that no interaction with the target has occurred. Thus, in the "no-interaction"-diagrams, the Wilson lines (represented by arrows) are gauge equivalent to the unit. Once the brackets in eq. (1) are multiplied out, keeping in mind that the transverse momentum is integrated over, we can observe the following simplification,

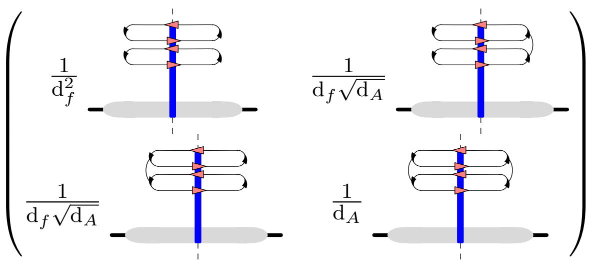


The first term in the above sum does not include an interaction with the target, and is thus of no particular interest to the discussion at hand. However, each of the terms in the curly braces includes one interaction: it is important to note that the $q\bar{q}$-dipole is explicitly in a singlet state before as well as after the interaction. Thus, the eikonal interaction in this situation did not destroy the "singlet-ness" of the dipole, even though each quark has experienced a color rotation. This feature of Wilson lines, that they leave singlets in tact, can also be observed in higher point correlators, [2].

In the case of a $q\bar{q}$-dipole, there exists only one possible singlet. For two $q\bar{q}$-dipoles, the situation becomes more interesting, since there are two possible singlets that can be formed, [1],

$\frac{1}{\mathrm{d}_f}$  and $\frac{1}{\sqrt{\mathrm{d}_A}}$ .

In an evolution equation such as the JIMWLK-equation, all of these singlets have to be considered simultaneously, as the Wilson line correlators may map one singlet into the other. Thus, the JIMWLK-evolution of two $q\bar{q}$-dipoles is governed by the following correlation matrix,

$\mathcal{A}(Y) :=$  (Y); (2)

the evolution of this correlation matrix is discussed in [1].

In order to determine the JIMWLK-evolution of any n-point correlator it is of utmost importance to know all possible singlet states for the corresponding Fock space constituents. In [3], we will present

(i) a counting argument determining the number of singlets

(ii) an algorithm which allows one to construct all singlets in a particular basis.

In this proceedings, we will illustrate the methods discussed in [3] by means of a concrete example.

The singlet operators of a particular configuration in Fock space do not represent independent degrees of freedom; we will exhibit this by means of an example in section 3. In order to effectively study the JIMWLK evolution of these operators, one usually performs a parameterization of the correlation matrix $\mathcal{A}(Y)$ as

$$\frac{\mathrm{d}}{\mathrm{d}Y}\mathcal{A}(Y) = \left(\frac{\mathrm{d}}{\mathrm{d}Y}\mathcal{A}(Y)\right)\mathcal{A}(Y)^{-1}\mathcal{A}(Y) =: -\mathcal{M}(Y)\mathcal{A}(Y),$$

provided $\mathcal{A}(Y)$ is invertible[1]. This equation can then be integrated to yield

$$\mathcal{A}(Y) = \mathcal{P}_Y \exp\left\{-\int_{Y_0}^{Y} \mathrm{d}Y'\mathcal{M}(Y')\right\}\mathcal{A}(Y_0),$$

for some initial rapidity Y_0. The Ansatz

$$\mathcal{M}(Y) = \frac{1}{2}\int \mathrm{d}\boldsymbol{u}\mathrm{d}\boldsymbol{v}\left(G_{Y,\boldsymbol{uv}}\bar{\nabla}^a_{\boldsymbol{u}}\bar{\nabla}^a_{\boldsymbol{v}} + \ldots\right),$$

where truncation after the first term is referred to as the *Gaußian Truncation*, [1], allows us to write each component (singlet) of $\mathcal{A}(Y)$ as an exponential with the functions G in the exponent. This parameterization therefore allows us to formulate the evolution equation in terms of elements of the tangent space rather than the singlet space itself, where the functions G now represent the *independent* degrees of freedom. However, to make sure that these independent degrees of freedom accurately represent the properties of the singlet algebra, the latter has to be studied in more detail. In this proceedings, we exhibit the fact that the Wilson line correlators are not independent degrees of freedom by means of an example. In particular, we will present all Wilson line correlators of the $3q+3\bar{q}$-algebra and take a coincidence limit between two Wilson lines, section 3. In future work, we will use coincidence limits such as the one presented in this paper in order to impose constraints on the functions G.

At this point, we note that the interesting behaviour of the n-point correlators in coincidence limits stems from the fact that the interaction is eikonal, that is localized and only via Wilson lines. A general n-point correlator for non-eikonal interactions does not exhibit any special behaviour when local coincidence limits are considered. For example, Figure 1 depicts a $q\bar{q}$-dipole interacting with a target t (indicated by the gray box) via the exchange of gluons:

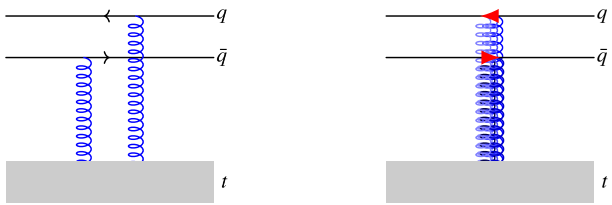


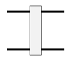
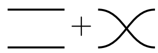
Figure 1. At low energies (*left* picture), the interaction of the $q\bar{q}$-dipole with the target t happens at no particular point in the x^--direction . At high energies (*right* picture), the target t is extremely Lorentz contracted, localizing the interaction at a particular point along the x^--axis.

[1] Since the correlation matrix becomes the unit matrix if all Wilson lines are set to unity, and one expects a smooth departure from unity as the Wilson lines smootly vary from $\mathbb{1}$, it is physically plausible to assume that $\mathcal{A}(Y)$ is invertible, at least in some neighbourhood of the unit matrix.

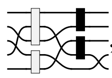
In a low-energy interaction therefore, a coincidence limit between the $q\bar{q}$-pair does not reveal new information, since the interaction between the dipole and the target is not localized in the x^--direction, unlike the high-energy counterpart.

2. Construction of general singlets and birdtrack notation

We assume a familiarity of the reader with the representation theory of $\mathsf{SU}(N_c)$ over an all-quark algebra by means of Young projection operators, see [4, 5] and other standard textbooks. This is known as the *theory of invariants*, which utilizes the the Young projection operators to classify all irreducible representations of $\mathsf{SU}(N_c)$. This topic recently received a more modern treatment when Cvitanović, [6], reformulated the theory of invariants in terms of birdtracks, which were originally developed by Penrose and MacCallum, [7]. In this formalism, symmetrizers (resp. anti-symmetrizers) are represented as empty (white) (resp. filled in (black)) boxes over the lines representing the tensor indices which they (anti-) symmetrize. For example,

$\boxed{1 \,|\, 2}$: $\boldsymbol{S}_{12}T^{ab} =$  $T^{ab} = \frac{1}{2}\Big($  $\Big) T^{ab} = \frac{1}{2}\left(T^{ab} + T^{ba}\right)$,

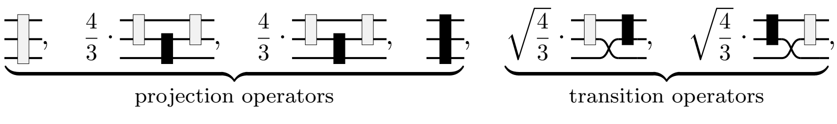
where $\boldsymbol{S}_{12}$ denotes the symmetrizer over the 1^{st} and 2^{nd} tensor index of T. In this way, the Young projection operator corresponding to the following Young tableau Θ can be written as

$$\Theta = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} \quad \mapsto \quad Y_\Theta = \alpha_\Theta \cdot$$  ,

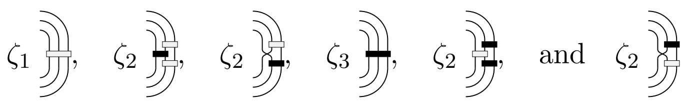
where Y_Θ denotes the Young projection operator corresponding to the Young tableau Θ, and α_Θ is the normalization constant needed to ensure the idempotency of Y_Θ, which can be obtained from the *hook length formula*, [6, 4, 8].

We will show in [3] that *any* singlet for *any* Fock-space configuration (such as $mq + \bar{m}\bar{q} + ng$) is equivalent to a singlet in the $kq + k\bar{q}$-algebra, where $k = k(N_c) \in \mathbb{N}$ is a N_c-dependent parameter, for a particular value of N_c. Therefore, it suffices to analyse the singlets of $\mathsf{SU}(N_c)$ over an algebra of the same number of quarks and anti-quarks.

The singlets of such an $mq + m\bar{q}$-algebra are obtained by *reshaping* the basis elements of the algebra of invariants of the mq-algebra, [3]. A particularly useful basis to choose for this process is that of Hermitean Young projection operators and transition operators, which we constructed in [9], as this basis will yield *orthogoinal* singlet states, [3]. For example, the Hermitean Young projection operators and transition operators for the $3q$-algebra are given by



giving the following singlet states for the $3q + 3\bar{q}$-algebra,



(3)

$$\text{with} \quad \zeta_1 = \frac{6}{(N_c+2)(N_c+1)N_c}, \quad \zeta_2 = \frac{3\cdot\theta(N_c-2)}{(N_c^2-1)} \quad \text{and} \quad \zeta_3 = \frac{6\cdot\theta(N_c-3)}{(N_c-2)(N_c-1)N_c},$$

where the top three lines denote the quark lines, and the bottom three lines denote the anti-quark lines.

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3. Wilson line correlators and coincidence limits

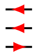
As mentioned previously, we are interested in the behaviour of Wilson line correlators. These are constructed as

$$\langle S_1|\mathbf{U}|S_2\rangle,$$

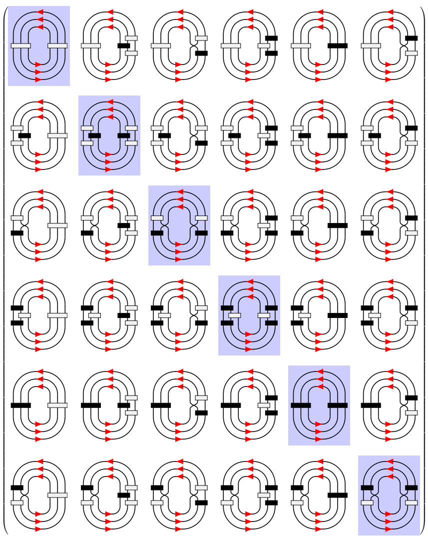
where $|S_1\rangle, |S_2\rangle$ are singlet states, $\langle S_1| = |S_1\rangle^\dagger$, and

$$\mathbf{U} = U_{\boldsymbol{x_1}} \otimes \ldots U_{\boldsymbol{x_n}} \otimes U^\dagger_{\boldsymbol{y_m}} \otimes \ldots U^\dagger_{\boldsymbol{y_1}}$$

is a tensor product of Wilson lines $U_{\boldsymbol{x_i}} \in \mathsf{SU}(N_c)$ with $\boldsymbol{x_i}$ being the x^+-coordinate of the i^{th} quark in the singlet. We will denote Wilson lines $U_{\boldsymbol{x_i}}$ by red arrowheads pointing from right to left and $U^\dagger_{\boldsymbol{y_j}}$ by red arrowheads pointing from left to right, suppressing the explicit coordinate dependence, for example

$$U_{\boldsymbol{x_1}} \otimes U_{\boldsymbol{x_2}} \otimes U^\dagger_{\boldsymbol{y_1}} \rightarrow$$  .

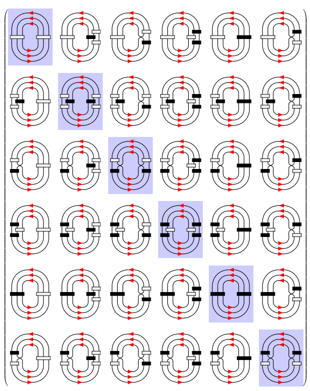
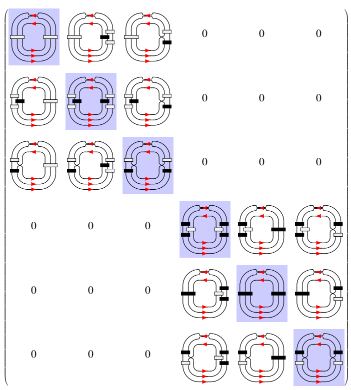
Returning to the $3q + 3\bar{q}$-example with singlet states (3), we can form a *correlation matrix* including each Wilson line singlet correlator as (where we have neglected the normalization factors for brevity)

$$\bar{\mathcal{A}}(Y) :=$$  .

When a coincidence limit between the Wilson lines acting on the top two quarks is taken, that is

$$U_{\boldsymbol{x_1}} \otimes U_{\boldsymbol{x_2}} \otimes U_{\boldsymbol{x_3}} \otimes U^\dagger_{\boldsymbol{y_3}} \otimes U^\dagger_{\boldsymbol{y_2}} \otimes U^\dagger_{\boldsymbol{y_1}} \xrightarrow[\boldsymbol{x_1}\rightarrow\boldsymbol{x_2}]{} U_{\boldsymbol{x_1}} \otimes U_{\boldsymbol{x_1}} \otimes U_{\boldsymbol{x_3}} \otimes U^\dagger_{\boldsymbol{y_3}} \otimes U^\dagger_{\boldsymbol{y_2}} \otimes U^\dagger_{\boldsymbol{y_1}},$$

then many of the Wilson line correlators in Eq (3) vanish, revealing a block-diagonal form, [3],

$$\bar{\mathcal{A}}(Y) =$$  $$\xrightarrow[\boldsymbol{x_1}\rightarrow\boldsymbol{x_2}]{}$$  . (4)

This example exhibits that the various Wilson line correlators in $\bar{\mathcal{A}}(Y)$ do not represent independent degrees of freedom - they vanish *simultaneously* as certain coincidence limits are considered. As was said previously, coincidence limits of correlation matrices such as in Eq. (4) are used to impose constraints on the functions G, which represent the independent degrees of freedom of the correlation matrix. It is clear that a complete set of constraints on the G's can however not be extracted from merely studying the coincidence limits of the singlet operators in one basis, but rather requires a study of these limits in *all* possible bases, and perhaps also an entirely different analysis of these singlets. In [3], we will give an expression of $\bar{\mathcal{A}}(Y)$ for the $3q + 3\bar{q}$-algebra in an alternate basis and discuss coincidence limits in this basis.

The algorithm giving all singlet states generates them in a particular basis, namely that of Hermitean Young projection operators and transition operators. While we do not (yet?) have a generic way of changing bases, having a counting argument for the dimension of the singlet algebra allows us to discern whether a different basis acquired by some other means is complete, and thus fit for an analysis of the singlet algebra.

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