Are we gauging the pressure correctly?

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Abstract. Thermodynamic properties of the quark-gluon plasma have been the subject of active investigation over the past decades. Monte Carlo lattice calculations have made great progress; in particular for the gluons, which form the first steps to full QCD. Nonetheless, 'artifacts' of this coarse-grained approach lead to uncertainties near the transition temperature, which are closely related to the divergences encountered in continuum perturbation theory. The latter must first be regularised and then renormalised, which we propose to do by comparing to the QCD trace anomaly. Fixing the analytic results at a semi-pertubative temperature, we find the bulk properties tend towards the free limit more gradually than has been presented in recent lattice findings.

1. Introduction

At sufficient energy densities, of about 1 GeV/fm^3, quantum chromodynamics (QCD) predicts that ordinary nuclear matter melts into a deconfined state of quarks and gluons. Heavy-ion collisions at RHIC and LHC are expected to probe this phase in the vicinity of the crossover temperature $T_c \sim 200$ MeV and part of the program to understand these experiments is pinpointing the equation of state. Although not physically realised, asymptotic freedom of QCD provides a rigorous limit for which the plasma is 'weakly coupled'. In this (high temperature) regime, well-defined approximations can be made and serve as a benchmark for theoretical statements.

The pressure, which is related to the thermodynamic potential $-pV$, is a quantity of central importance for statistical physics. Knowing the temperature dependence, one derives the entropy density $s = \partial p/\partial T$ and energy density $e = Ts - p$. Lattice gauge theory has been the long-standing approach to calculate equilibrium properties of QCD near T_c, and has revealed that interactions remain 'strong' for $T_c < T \lesssim 4T_c$. Recent studies in the quenched limit have obtained the equation of state for values $T \in [0.7, 1000]T_c$, thus affording a comparison with perturbation theory [1]. For an ideal gluon gas, the Stefan-Boltzmann pressure is

$$p_{\text{SB}} = d_g \frac{\pi^2}{90} T^4 \,,$$

where $d_g = 16$ is the gluon degeneracy factor.

Interactions, built into QCD via the parameter α make the pressure deviate from its free value. By virtue of the momentum dependence of the coupling constant $\alpha(q)$ for large arguments ($q \gg \Lambda$, which we define later), perturbation theory can be used. This remains true at high temperatures, where the average momentum $\bar{q}$ is similar to the temperature, $\bar{q} \sim T$ (in equilibrium). Let the pressure be represented as an expansion in powers of α, with

$$p(T;\,\alpha) \;=\; p_0(T) + p_1(T) + p_2(T) + \dots\,, \tag{1}$$

where $p_n(T) \sim \mathcal{O}(\alpha^n)$. The normalisation is set by $p_0(T) = p(T;\,0) = p_{\rm SB}$ and yields an upper bound $p_{\rm SB} \geq p(T;\,\alpha)$.

The lowest order $\mathcal{O}(\alpha)$ diagrams give a negative correction to the pressure, which may also be related to an 'excluded volume' for gluons. For certain graphs of order $\mathcal{O}(\alpha^n)$, $n \geq 2$, the direct evaluation gives $p_n = \infty$. Such *non-analytic* behaviour stems from the long ranged nature of the (gauge) force, and is well known from the classical Coulomb gas [2]. The remedy is to resum infinitely many diagrams, giving a term of order $\mathcal{O}(\alpha^{3/2})$ in Eq.(1). After this, at order $\mathcal{O}(\alpha^2)$, a term $\alpha^2\log(\alpha)$ occurs and requires 3-loop diagrams to be screened [3]. Furthermore, the $\mathcal{O}(\alpha^2)$ formula compels the use of the coupling $\alpha(\mu)$, where μ is a renormalisation scale appearing in a manner such that $\partial p\,(T;\,\alpha(\mu))\,/\partial\mu$ is higher order than the approximation used.

The perturbative expansion for $p(T;\,\alpha)$ is known through order $\mathcal{O}(\alpha^{5/2})$, beyond which the serious infrared difficulties of QCD prohibit further progress [4]. Nevertheless, asymptotic formulae can be useful when truncated to a finite number of terms. A *fixed order* calculation becomes precise[1] in the limit $\alpha \to 0$, for which purpose we shall study

$$p\Big(T;\,\alpha(2\pi T)\Big) \;=\; p_{\rm SB}\,\Big\{1 + c_1\alpha + c_{3/2}\alpha^{3/2} + \alpha^2(c_2 + c_2'\log(\alpha)) + c_{5/2}\alpha^{5/2} + \mathcal{O}(\alpha^3)\Big\}\,. \tag{2}$$

The term $\mathcal{O}(\alpha^3)$ contains a last perturbative piece, see [5], and a non-perturbative coefficient. Since the asymptotic expansion does not converge, it is particularly sensitive to this last coefficient – which has been 'fit' to the lattice data [1]. To avoid this aspect, we simply use Eq. (2), with the *known* coefficients (and the canonical scale $\mu = 2\pi T$), to argue our case.

2. Renormalising via $w(T)$

Although $p(T)$ may be obtained on the lattice [6], it is more convenient to calculate the *interaction measure* (also called trace anomaly),

$$w(T) \;:=\; e(T) - 3p(T) \;=\; \mathrm{Tr}\,\Theta\,, \tag{3}$$

where $\Theta^{\mu\nu}(T)$ is the energy momentum tensor, equal to $\mathrm{diag}(e,p,p,p)$ in the local rest frame. It thus follows that $w(T) \geq 0$ and this is indeed met by (2), reproducing the fact that $p(T)/p_{\rm SB}$ is a smoothly *increasing* function of T.

For a conformal theory, the equation of state $e(T) = 3p(T)$ is true even for fully interacting systems. Therefore, $w(T) \neq 0$ can only arise for a system an with an additional scale, other than T and V. This is indeed the case for QCD, where the energy scale Λ enters as a pole in α, for example the 2-loop formula

$$\alpha(\mu) \simeq \frac{4\pi}{11t}\Big\{\,1 - \frac{102\log t}{121t}\,\Big\}\,, \quad \text{with} \quad t = \log(\mu^2/\Lambda^2)\,.$$

To clarify this point, note that from (3) we may express

$$w(T) \;=\; T\frac{d}{dT}\Big\{\,\frac{p(T)}{T^4}\,\Big\}\,. \tag{4}$$

Hence, for a 'fixed' coupling, $p(T;\,\alpha)/T^4$ is constant and thus $w(T) = 0$. In order to obtain a non-zero $w(T)$, it is necessary to renormalise, i.e. specify how α in Eq.(2) depends on T. To

[1] Here we mean that the absolute error tends to zero, usually as $\sim \exp(-1/\alpha)$, although the sign may (and does in this case) depend on the level of truncation.

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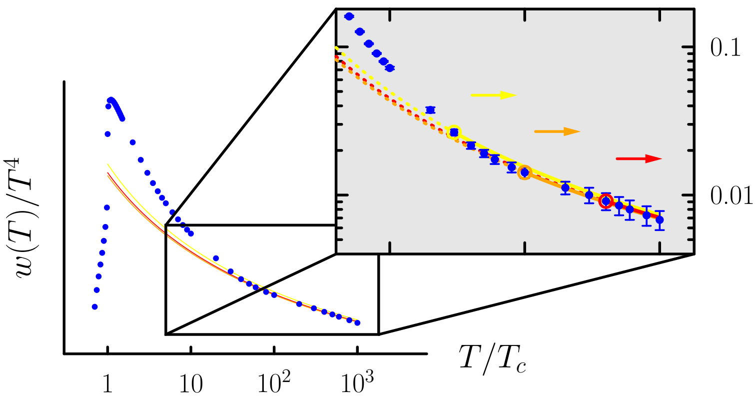

Figure 1. The interaction measure as a function of temperature as calculated on the lattice [1]. The inset focuses on $T \gtrsim 5T_c$, and shows the perturbative result for $w(T)$ with the renormalisation points circled in yellow, orange and red ($T_\star = \{30, 100, 400\}T_c$ respectively). For $T > T_\star$ as shown by the arrows, the formulae consistently agrees with the lQCD data.

this end, we choose the popular scale $\mu = 2\pi T$ and use the 2-loop formula for the coupling. Applying (4) to Eq.(2) yields a model for $w(T)$. Our idea is then to fix the residual scale Λ, by directly matching the perturbative formula to the lattice value, viz. $w(T_\star) = w_\star^{\mathrm{lQCD}}$. Since the lattice results are calculated in units of T_c, the single free parameter in our comparison is the ratio $\lambda = \Lambda/T_c$.

Fig. 1 shows the results for $w(T)$, using a semi-perturbative $T_\star$ to fix the value of λ. Testing the scheme at $T_\star = \{30, 100, 400\}T_c$ consistently gives $\lambda \simeq 0.5$. Provided that $\alpha(T_\star)$ is sufficiently small[2] at the determined value for λ, approximations for $w(T)$, where $T \gg T_\star$, should (and do) corroborate the lattice values. Below $T_\star$, there is no reason to expect that $p(T;\alpha)$ should converge and the shape of $w(T)$ for temperatures $T \gtrsim 2T_c$ has been understood in terms of quasiparticles [7]. The pressure in Eq.(2) is then specified, having adjusted λ in our scheme to w^{lQCD}.

One of the principle challenges in lattice QCD, is taking the continuum limit (which is, not by coincidence, the same limit that Λ emerges). By computing $w(T)$, one avoids having to subtract the zero point contribution to $p(T)$ or $e(T)$. From the interaction measure (4), given as a function of T, the pressure can be reconstructed, up to an integration constant,

$$\frac{p(T)}{T^4} = \frac{p(T_0)}{T_0^4} + \int_{T_0}^{T} \frac{d\tau}{\tau} \left\{ \frac{w(\tau)}{\tau^4} \right\}. \tag{5}$$

This integral method depends on $w(T)$ over a range of values for T. In particular, the area bounded by the interaction measure over all temperatures gives the normalisation of $p(T)$ [presuming that $p(T) \to 0$ at zero temperature]. With information on $w(T)$ only at a discrete set of values for T, Eq.(5) is at best a Riemann sum, and dependent on extrapolation to 'endpoints' $T = 0$ or $T = \infty$, depending on whether $T \gtrless T_0$. Furthermore, $p(T)/T^4$ receives a large numerical contribution near the peak value of $w(T)/T^4$, at about $T/T_c \simeq 1.1$.

[2] Characterising α as 'small' depends on the nature of the asymptotic series.

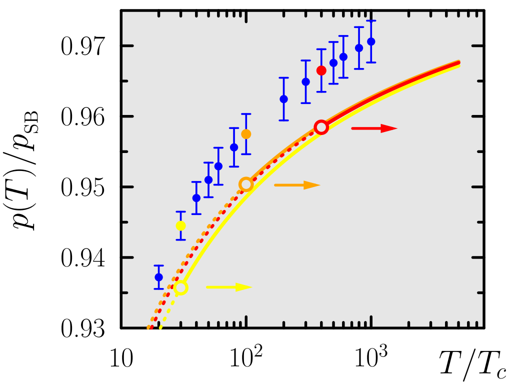

Figure 2. The pressure as a function of temperature (in units of the free value) for $n_f = 0$. Shown as blue points are the lattice data [1]. The yellow, red and orange curves are obtained by renormalising according to $w(T)$, Fig. 1. Solid lines indicate $T_\star < T$, where the approximation schemes *should* converge. Evidently there is a systematic discrepancy of $\sim 1\%$ with the lattice data.

3. Reason for concern

Given the precision of pure gauge lQCD data, we now point out a discrepancy that has so far escaped notice. The previous section outlined two approaches to calculating the pressure $p(T)$,

(i) using a truncated asymptotic formula, i. e. Eq.(2),

(ii) by means of (5), from $w(T)$ determined on the lattice [1].

Shown in Fig. 2 is $p(T)$ at large temperatures $T > 10T_c$ according to these two distinct methods. Evidently they disagree at about 1% of the free limit and, more urgently, the values for $p(T_\star)$ fail to match at the renormalisation point. Method (i) appears to give robust predictions for $T_\star < T$, but systematically *underestimates* the lattice values.

This failure is actually crucial to resolve because it emerges from a well-defined limit, in which theoretical descriptions *should* be under control. The perturbative analysis of the QCD trace anomaly indicates a slower approach of the $p(T)$ to the free limit [and similarly for $e(T)$ and $s(T)$]. Our conclusion may seem of little consequence for phenomonology of heavy-ion collisions. However, an uncertainty of one percent in $p(T)/p_{\mathrm{SB}}$ can translate into an order of magnitude in the temperature T/T_c (see Fig. 2). This casts aspersions on hydrodynamic simulations, where the equation of state is needed to evolve $\Theta^{\mu\nu}$ prior to freeze-out.

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