Integrability conditions for nonrotating solutions in $f(R)$ gravity

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Abstract. Several classes of cosmological models with irrotational fluid flows and where the underlying theory of gravitation is $f(R)$-gravity are investigated. The integrability conditions describing a consistent evolution of the linearized field equations of shear-free dust universes are presented. We also derive consistency relations of models with more severe constraints, such as non-expanding spacetimes as well as those spacetimes with vanishing gravito-magnetic or gravito-electric components of the Weyl tensor.

1. Introduction

Among the most common generalizations to the General Theory of Relativity (GR) to explain current deficits in the energy budget of the universe, and hence to explain cosmic acceleration, are higher-order theories of gravity. Models that include functions of the Ricci curvature R in the Hilbert-Einstein action

$$\mathcal{A} = \tfrac{1}{2}\int d^4x\sqrt{-g}\,[f(R) + 2\mathcal{L}_m]\ , \tag{1}$$

where $\mathcal{L}_m$ is the matter field Lagrangian, and result in fourth-order field equations are referred to as $f(R)$-gravity theories [1, 2, 3, 4]. The generalized field equations arising from such action, obtained using the standard variational principle with respect to the metric g_{ab}, can be represented by

$$f'G_{ab} = T^m_{ab} + \frac{1}{2}(f - Rf')g_{ab} + \nabla_b\nabla_a f' - g_{ab}\nabla_c\nabla^c f'\ , \tag{2}$$

where G_{ab} and T^m_{ab} are the standard notations for the Einstein tensor and the energy-momentum tensor of standard matter. The extra terms on the right-hand side account for what is called the curvature-fluid energy-momentum tensor and identically vanish in GR. Here f is a shorthand for the $f(R)$ function and primes indicate derivatives with respect to R.

We assume the universe is filled with standard matter and curvature fluid sources and its total energy density, isotropic pressure, anisotropic pressure and heat flux terms are given, respectively, by [5, 6]

$$\mu \equiv \frac{\mu_m}{f'} + \mu_R\ , \quad p \equiv \frac{p_m}{f'} + p_R\ , \qquad \pi_{ab} \equiv \frac{\pi^m_{ab}}{f'} + \pi^R_{ab}\ , \quad q_a \equiv \frac{q^m_a}{f'} + q^R_a\ , \tag{3}$$

where μ_m and μ_R stand, respectively, for the energy density of standard matter and curvature fluids, etc.

To linear-order perturbations around a Friedmann-Lemaître-Robertson-Walker (FLRW) background, the curvature fluid component are defined as

$$\mu_R = \frac{1}{f'}\left[\tfrac{1}{2}(Rf'-f) - \Theta f''\dot{R} + f''\tilde{\nabla}^2 R\right] , \quad (4)$$

$$p_R = \frac{1}{f'}\left[\tfrac{1}{2}(f-Rf') + f''\ddot{R} + f'''\dot{R}^2 + \tfrac{2}{3}\left(\Theta f''\dot{R} - f''\tilde{\nabla}^2 R\right)\right] , \quad (5)$$

$$q_a^R = -\frac{1}{f'}\left[f'''\dot{R}\tilde{\nabla}_a R + f''\tilde{\nabla}_a\dot{R} - \tfrac{1}{3}f''\Theta\tilde{\nabla}_a R\right] , \quad (6)$$

$$\pi_{ab}^R = \frac{f''}{f'}\left[\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R - \sigma_{ab}\dot{R}\right] , \quad (7)$$

where the overdot $\dot{}$ and $\tilde{\nabla}$ indicate time and covariant spatial derivatives. In the $1+3$-covariant decomposition formalism, fundamental observers with 4-velocity vectors u^a slice spacetime into constant time and space hypersurfaces. We use u^a to define covariant time derivatives and the projection tensor $h_{ab} = u_a u_b + g_{ab}$ is used is used to define the fully orthogonally projected covariant derivative of tensors. We denote the orthogonally projected symmetric trace-free part of vectors and rank-2 tensors as

$$V^{\langle a\rangle} = h^a_b V^b \ , \ S^{\langle ab\rangle} = \left[h^{(a}{}_c h^{b)}{}_d - \tfrac{1}{3}h^{ab}h_{cd}\right] S^{cd} , \quad (8)$$

and the volume element for the rest spaces orthogonal to u^a is given by [7]

$$\varepsilon_{abc} = u^d \eta_{dabc} = -\sqrt{|g|}\delta^0{}_{[a}\delta^1{}_b\delta^2{}_c\delta^3{}_{d]}u^d \Rightarrow \varepsilon_{abc} = \varepsilon_{[abc]},\ \varepsilon_{abc}u^c = 0, \quad (9)$$

where η_{abcd} is the 4-dimensional volume element with the properties $\eta_{abcd} = \eta_{[abcd]} = 2\varepsilon_{ab[c}u_{d]} - 2u_{[a}\varepsilon_{b]cd}$. In this work, brackets (ab) and square brackets $[ab]$ denote symmetrization and anti-symmetrization over the indices a and b. Covariant spatial divergence and curl of tensors are given as

$$\text{div}V = \tilde{\nabla}^a V_a \,, \qquad (\text{div}S)_a = \tilde{\nabla}^b S_{ab} \,, \quad (10)$$

$$\text{curl}\, V_a = \varepsilon_{abc}\tilde{\nabla}^b V^c \,, \quad \text{curl}\, S_{ab} = \varepsilon_{cd(a}\tilde{\nabla}^c S_{b)}{}^d \,. \quad (11)$$

The full covariant derivative of u^a can be split into its irreducible parts as

$$\nabla_a u_b = -A_a u_b + \tfrac{1}{3}h_{ab}\Theta + \sigma_{ab} + \varepsilon_{abc}\omega^c, \quad (12)$$

where $A_a \equiv \dot{u}_a$, $\Theta \equiv \tilde{\nabla}_a u^a$, $\sigma_{ab} \equiv \tilde{\nabla}_{\langle a}u_{b\rangle}$ and $\omega^a \equiv \varepsilon^{abc}\tilde{\nabla}_b u_c$ are the acceleration, expansion, shear and vorticity (rotation) of the fluid flow. The *Weyl conformal curvature tensor* C_{abcd} is defined from the Riemann tensor R^a_{bcd}as

$$C^{ab}{}_{cd} = R^{ab}{}_{cd} - 2g^{[a}{}_{[c}R^{b]}{}_{d]} + \frac{R}{3}g^{[a}{}_{[c}g^{b]}{}_{d]} \quad (13)$$

and can be split into its "gravito-electric" and "gravito-magnetic" parts, respectively, as

$$E_{ab} \equiv C_{agbh}u^g u^h, \qquad H_{ab} = \tfrac{1}{2}\eta_{ae}{}^{gh}C_{ghbd}u^e u^d. \quad (14)$$

E_{ab} and H_{ab} represent the free gravitational field, enabling gravitational action at a distance (tidal forces and gravitational waves), and influence the motion of matter and radiation through the geodesic deviation for timelike and null vector fields, respectively.

The linearised evolution equations in $f(R)$ gravity are given by [5, 8]:

$$\dot{\mu}_m = -(\mu_m + p_m)\Theta - \tilde{\nabla}^a q^m_a \ , \tag{15}$$

$$\dot{\mu}_R = -(\mu_R + p_R)\Theta + \frac{\mu_m f''}{f'^2}\dot{R} - \tilde{\nabla}^a q^R_a \ , \tag{16}$$

$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 - \tfrac{1}{2}(\mu + 3p) + \tilde{\nabla}_a A^a \ , \tag{17}$$

$$\dot{q}^m_a = -\tfrac{4}{3}\Theta q^m_a - \mu_m A_a \ , \tag{18}$$

$$\dot{q}^R_a = -\tfrac{4}{3}\Theta q^R_a + \frac{\mu_m f''}{f'^2}\tilde{\nabla}_a R - \tilde{\nabla}_a p_R - \tilde{\nabla}^b \pi^R_{ab} \ , \tag{19}$$

$$\dot{\omega}_a = -\tfrac{2}{3}\Theta\omega_a - \tfrac{1}{2}\varepsilon_{abc}\tilde{\nabla}^b A^c \ , \tag{20}$$

$$\dot{\sigma}_{ab} = -\tfrac{2}{3}\Theta\sigma_{ab} - E_{ab} + \tfrac{1}{2}\pi_{ab} + \tilde{\nabla}_{\langle a}A_{b\rangle} \ , \tag{21}$$

$$\dot{E}_{ab} + \tfrac{1}{2}\dot{\pi}_{ab} = \varepsilon_{cd\langle a}\tilde{\nabla}^c H^d_{b\rangle} - \Theta E_{ab} - \tfrac{1}{2}(\mu + p)\,\sigma_{ab} - \tfrac{1}{2}\tilde{\nabla}_{\langle a}q_{b\rangle} - \tfrac{1}{6}\Theta\pi_{ab} \ , \tag{22}$$

$$\dot{H}_{ab} = -\Theta H_{ab} - \varepsilon_{cd\langle a}\tilde{\nabla}^c E^d_{b\rangle} + \tfrac{1}{2}\varepsilon_{cd\langle a}\tilde{\nabla}^c \pi_{b\rangle}{}^d \ , \tag{23}$$

and propagate consistent initial data on some initial hypersurface S_0 uniquely along the (generally future-directed) reference timelike congruence. The initial conditions to be specified for the above evolution equations are restricted by the constraint equations

$$(C^1)_a := \tilde{\nabla}^b\sigma_{ab} - \tfrac{2}{3}\tilde{\nabla}_a\Theta + \varepsilon_{abc}\tilde{\nabla}^b\omega^c + q_a = 0 \ , \tag{24}$$

$$(C^2)_{ab} := \varepsilon_{cd(a}\tilde{\nabla}^c\sigma_{b)}{}^d + \tilde{\nabla}_{\langle a}\omega_{b\rangle} - H_{ab} = 0 \ , \tag{25}$$

$$(C^3)_a := \tilde{\nabla}^b H_{ab} + (\mu + p)\omega_a + \tfrac{1}{2}\varepsilon_{abc}\tilde{\nabla}^b q^c = 0 \ , \tag{26}$$

$$(C^4)_a := \tilde{\nabla}^b E_{ab} + \tfrac{1}{2}\tilde{\nabla}^b\pi_{ab} - \tfrac{1}{3}\tilde{\nabla}_a\mu + \tfrac{1}{3}\Theta q_a = 0 \ , \tag{27}$$

$$(C^5) := \tilde{\nabla}^a\omega_a = 0 \ , \tag{28}$$

$$(C^6)_a := \tilde{\nabla}_a p_m + (\mu_m + p_m)A_a = 0 \tag{29}$$

which must remain satisfied on any hypersurface Σ_t for consistency of the field equations.

2. Consistency analysis of irrotational spacetimes

Irrotational fluid flows have vanishing vorticity ($\omega_a = 0$). Imposing this vanishing vorticity condition on the evolution equations (15)-(23) results in Eq. (20) turning into a new constraint

$$(C^{6*})_a := \varepsilon_{abc}\tilde{\nabla}^b A^c = 0 \implies A_a = \tilde{\nabla}_a\psi \text{ for some scalar } \psi \ . \tag{30}$$

To check for temporal consistency, we propagate this constraint to obtain

$$\left(\varepsilon_{abc}\tilde{\nabla}^b A^c\right)^{\cdot} = 0 \ , \tag{31}$$

which is an identity. Let us now take the curl of this constraint to check for spatial consistency:

$$curl(curl(A_a)) = \tilde{\nabla}_a\left(\tilde{\nabla}^2\psi\right) - \tilde{\nabla}^2\left(\tilde{\nabla}_a\psi\right) + \tfrac{2}{3}\left(\mu - \tfrac{1}{3}\Theta^2\right)\tilde{\nabla}_a\psi = 0 \ , \tag{32}$$

which is also an identity because for any scalar and vector field ϕ and V_a

$$\tilde{\nabla}^2\left(\tilde{\nabla}_a\psi\right) = \tilde{\nabla}_a\left(\tilde{\nabla}^2\psi\right) + \tfrac{1}{3}\tilde{R}\tilde{\nabla}_a\psi \ , \tag{33}$$

$$\text{curl}(\text{curl}V_a) = \tilde{\nabla}_a\left(\tilde{\nabla}^b V_b\right) - \tilde{\nabla}^2 V_a + \tfrac{2}{3}\left(\mu - \tfrac{1}{3}\Theta^2\right)V_a \ . \tag{34}$$

2.1. Shear-free dust spacetimes

Pure dust spacetimes are characterised by $w = 0 = p_m$, $q^m_a = 0 = A_a$, $\pi^m_{ab} = 0$, and shear-free models are fluid flow models with $\sigma_{ab} = 0$. For such models, Eq. (21) turns into a new constraint

$$(C^{5d})_{ab} := E_{ab} - \tfrac{1}{2}\pi^R_{ab} = 0 \ . \tag{35}$$

the temporal and spatial consistencies of which have to be checked. Unlike for shear-free dust spacetimes in GR, the electric component of the Weyl tensor does not vanish because of the non-vanishing contribution of the anisotropic pressure π^R_{ab}. But Eq. (25) shows that H_{ab} does vanish, leading to a modified constraint from Eq. (23)

$$(C^{6d})_{ab} := \varepsilon_{cd\langle a}\tilde{\nabla}^c E^d_{b\rangle} - \tfrac{1}{2}\varepsilon_{cd\langle a}\tilde{\nabla}^c \pi^{d\ d}_{b\rangle} = 0 \ , \tag{36}$$

which is an identity by virtue of Eq. (35). Moreover, Eq. (26) shows that q^R_a is irrotational and can, therefore, be written as the gradient of a some scalar field ϕ:

$$q^R_a = \tilde{\nabla}_a \phi \ . \tag{37}$$

Since from (24), $q^R_a = \frac{2}{3}\tilde{\nabla}_a\Theta$ we have, for irrotational and shear-free dust spacetimes,

$$\phi = \tfrac{2}{3}\Theta + C \ , \tag{38}$$

for some spatially constant scalar C. Using Eq. (6) in (37), an interesting integrability condition is obtained:

$$\tfrac{2}{3}f'\tilde{\nabla}_a\Theta + \left(f'''\dot{R} - \tfrac{1}{3}\Theta f''\right)\tilde{\nabla}_a R + f''\tilde{\nabla}_a\dot{R} = 0 \ . \tag{39}$$

In the GR limit, i.e., $f = R$, $f' = 1$, $f'' = f''' = 0$, the above consistency relation leads to $\tilde{\nabla}_a\Theta = 0$, which is trivially true for the class of models under consideration. To check for temporal consistency of Eq. (35), let us take the time derivative of both sides of this equation to obtain

$$\dot{\pi}^R_{ab} + \tfrac{2}{3}\Theta\pi^R_{ab} - \tfrac{1}{2}\tilde{\nabla}_{\langle a}q^R_{b\rangle} = 0 \ , \tag{40}$$

which, using Eqs. (6) and (7), yields

$$\left[\frac{3}{2}\left(\frac{f'''}{f'} - \frac{f''^2}{f'^2}\right)\dot{R} - \frac{\Theta f''}{6f'}\right]\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R + \tfrac{3f''}{2f'}\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}\dot{R} = 0 \ . \tag{41}$$

This implies that *irrotational shear-free dust spacetimes governed by $f(R)$ gravitational physics evolve consistently if Eq. (41) is satisfied.* Note that the GR limit of this equation is an identity since the left-hand side vanishes identically. Now the curl of the above equation gives

$$\left[\frac{3}{2}\left(\frac{f'''}{f'} - \frac{f''^2}{f'^2}\right)\dot{R} - \frac{\Theta f''}{6f'}\right]\varepsilon_{cda}\tilde{\nabla}^c\tilde{\nabla}_{\langle b}\tilde{\nabla}^{d\rangle}R + \tfrac{3f''}{2f'}\varepsilon_{cda}\tilde{\nabla}^c\tilde{\nabla}_{\langle b}\tilde{\nabla}^{d\rangle}\dot{R} = 0 \ , \tag{42}$$

which is an identity since, for any scalar field ψ,

$$\varepsilon_{cda}\tilde{\nabla}^c\tilde{\nabla}_{\langle b}\tilde{\nabla}^{d\rangle}\psi = \varepsilon_{cda}\tilde{\nabla}^c\tilde{\nabla}_{(b}\tilde{\nabla}^{d)}\psi = \varepsilon_{cda}\tilde{\nabla}^c\tilde{\nabla}_b\tilde{\nabla}^d\psi = 0 \ . \tag{43}$$

Thus, *all irrotational shear-free dust spacetimes in $f(R)$-gravity are consistent.*

If we make a further restriction and turn off E_{ab}, a locally conformally flat metric is obtained. For this class of models a new linearized constraint emerges from Eq. (22):

$$\tilde{\nabla}_{\langle a}q^R_{b\rangle} = 0 = \left(\dot{R}f''' - \tfrac{1}{3}\Theta f''\right)\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R + f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}\dot{R} \ , \tag{44}$$

and from Eq. (21) we get

$$\pi^R_{ab} = 0 = f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R \,. \tag{45}$$

For GR, $f'' = 0$ and an identity results. For $f'' \neq 0$, the equation leads to the constraint on the Ricci curvature: $\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R = 0$. Using this and the relation

$$\left(\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}\psi\right)^{\cdot} = \tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}\dot{\psi} - \tfrac{2}{3}\Theta\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}\psi + \dot{\psi}\tilde{\nabla}_{\langle a}A_{b\rangle} \,, \tag{46}$$

in Eq. (44) leads to an identity. As a result, *linearised $f(R)$ field equations in irrotational and shear-free dust spacetimes with vanishing Weyl tensor are consistent.*

2.2. Dust models with divergence-free H_{ab}
A necessary condition for gravitational radiation is the the vanishing of the divergence of a non-zero H_{ab}. If we prescribe this condition on the field equations, Eq. (38) generalizes to

$$\tilde{\nabla}_a\phi = \tfrac{2}{3}\tilde{\nabla}_a\Theta - \tilde{\nabla}^b\sigma_{ab} \,. \tag{47}$$

A subclass of such models, called "purely radiative" dust spacetimes, is a divergence-free E_{ab}. Such models in $f(R)$ gravity are constrained further as

$$\tilde{\nabla}_a\mu_m + f'\tilde{\nabla}_a\mu_R + f'\Theta q^R_a - \tfrac{3f'}{2}\tilde{\nabla}^b\pi^R_{ab} = 0 \tag{48}$$

as a result of Eq. (27). Purely radiative irrotational dust spacetimes in GR should be spatially homogeneous (with $\tilde{\nabla}_a\mu_m = 0$).

Models with vanishing gravito-electric component of the Weyl tensor are referred to as *anti-Newtonian* models because they are considered to be the most extreme of non-Newtonian gravitational models [9, 10, 11]. Although there are no anti-Newtonian solutions of linearized perturbations of FLRW in GR, it has been shown that such restrictions are conditional (of integrability conditions) in $f(R)$ gravity [11].

2.3. Non-expanding spacetimes
Here we want to explore the (in)consistencies that emerge assuming theoretical cases of a nonexpanding spacetime, i.e., $\Theta = 0$. One gets the evolution equation for matter heat flux

$$\dot{q}^m_a = \frac{w}{1+w}\tilde{\nabla}_a\mu_m \,, \tag{49}$$

and a new constraint arises from the Raychaudhuri equation (17)

$$(C^{6s}) := \tilde{\nabla}_aA^a - \tfrac{1}{2f'}(1+3w)\mu_m - \tfrac{1}{2}\left(\mu_R + 3p_R\right) = 0 \,. \tag{50}$$

It follows that dust models ($A_a = 0 = q^m_a$) have active gravitational mass $\mu + 3p = 0$. Since (15) implies $\mu_d(t) = \text{const}$, we notice that $\mu_R + 3p_R = \text{const}$, as well. From the definitions (4) and (5) for μ_R and p_R and the *trace equation*

$$3f''\ddot{R} + 3\dot{R}^2f''' + 3\Theta\dot{R}f'' - 3f''\tilde{\nabla}^2R - Rf' + 2f - \mu_m + 3p_m = 0 \,, \tag{51}$$

we conclude that (2.3) implies

$$f - 2f''\tilde{\nabla}^2R = \text{const} \,. \tag{52}$$

Thus any nonrotating and noexpanding dust spacetime in $f(R)$ cosmology should have a gravitational Lagrangian that satisfies Eq. (52).

3. Conclusion

We have looked at the consistency relations of linearized perturbations of FLRW universes with irrotational fluid flows arising as a result of imposing special restrictions to the field equations. We have shown that linearized shear-free dust models have a vanishing gravito-magnetic component of the Weyl tensor. The case of vanishing full Weyl tensor in linearised $f(R)$ field equations has also been explored, as well as those models with purely gravito-magnetic spacetimes. A subclass of gravito-magnetic models are those in which the divergence of H_{ab} is zero, a necessary condition for emission of gravitational waves. In GR, it is known that these models are homogeneous dust FLRW universes. We have shown that the homogeneity condition is not necessary in $f(R)$ gravity. Lastly, we have derived an integrability condition for non-rotating and no-expanding dust spacetimes in $f(R)$ gravity.

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