NLO Rutherford Scattering and Energy Loss in a QGP

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Abstract.
We calculate to next-to-leading order the cross section of a massless electron scattered off of a static point charge in the $\overline{\text{MS}}$ renormalization scheme. Since we use the $\overline{\text{MS}}$ renormalization scheme, our result is valid up to arbitrary large momentum transfers between the source and the scattered electron. We then investigate the importance of the BN vs. KLN theorems in various theories as we work towards computing the NLO corrections to the energy loss of a QCD particle propagating in a quark-gluon plasma.

1. Introduction

The Quark Gluon Plasma (QGP) is believed to be the state of matter in the first few microseconds after the Big Bang [1, 2]. The QGP has been predicted to exist by the Quantum Chromodynamics (QCD) at a very high energy density and very high temperature (~ 180 MeV). The temperature dependence of the energy density in QCD is one of the results of the lattice QCD [3, 4]. Which shows a rapid change of the energy density at the critical temperature (T_c). This rapid change has been interpreted as the change of degrees of freedom in the system. Well below T_c, there are three hadronic degrees of freedom due to the three lightest hadrons: π^+, π^- and π^0. Well above T_c, there are $2(N_c^2 - 1) + 2 \times 2 \times N_c \times N_f$ degrees of freedom from the fundamental gluons and quarks of the theory.

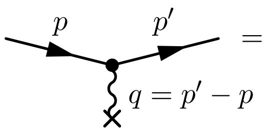
Studying the high $p_\perp$ interactions at RHIC and LHC shows that the jet quenching is due to the final state energy loss. The leading-order pQCD calculations give a good estimate for the energy loss [5,6]. The question now is, what do we expect to find if we include the next-to-leading (NLO) contributions? We wish to check the self-consistency of these pQCD results and to make the pQCD calculation more quantitative. As a first step towards the NLO pQCD calculations, we calculate in this paper the NLO corrections to the elastic scattering of a massless electron scattered off of a static source.

2. The leading term of the scattering cross section

We consider the Lagrangian describing an electron scattered off of a classical source $J^\mu(x) = V^\mu \delta^{(4)}(\vec{x} - \vec{V}x^0)$, where V^μ is the unit time-like velocity vector

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi} \left(i\partial\!\!\!/ - m \right) \psi - e\bar{\psi}\gamma^\mu \psi A_\mu + eJ_\mu A^\mu, \tag{1}$$

The Feynman rules for this Lagrangian will be exactly the same as in the normal QED Lagrangian [7] in addition to that for each external source we write $-ieV^{\mu}$. Let p and p' to be the momenta of the incoming and the outgoing electrons respectively. The delta function from the source $J^{\mu}(x)$ ensures the conservation of energy $E_{p'} = E_p = E$ and that the momentum transfer to be $q = p' - p$. Now we write down the amplitude of the leading term using the Feynman rules

$i\mathcal{M}_0 =$ 

$$= \frac{i\,e^2}{q^2}\overline{u}^{s'}(p')\,\gamma^0\,u^s(p). \tag{2}$$

We recall the identity $\sum_s u^s(p)\,\bar{u}^s(p) = \not{p} + m$, the trace technology and the properties of the γ-matrices. The leading term of the differential cross section will be

$$\left(\frac{d\sigma}{d\Omega}\right)_0 = \frac{2\alpha^2}{q^4}\left(2E^2 - p\cdot p'\right), \tag{3}$$

where we set the mass of the electron to be zero.

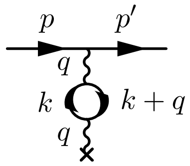
3. Renormalization of the Lagrangian

The NLO diagrams usually contain either fermion or photon loops. These loops require integrations over the loop momentum which usually diverge in 4-dimensions. In order to remove the divergences from our calculations, we first define the divergent parts by using the dimensional regularization to regularize the UV divergences and the mass regularization for the IR divergences. Then we renormalize the Lagrangian using the systematic way of renormalization [7], then we apply the renormalization scheme to get rid of the UV divergences. In this paper, we use $\overline{\text{MS}}$ renormalization scheme to tame the UV divergences as we are dealing with a massless theory [8]. We will follow the calculations of the differential cross section with massive electron while using $\overline{\text{MS}}$ allows us to set $m_e = 0$ safely.

The Dimensional regularization requires replacing the 4-momentum integral by an integral over the momentum in d-dimensions. Which also requires rescaling the electron charge e by the factor $\mu^{\frac{4-d}{2}}$, where μ is any mass scale to ensure that e remains dimensionless [9]. At the end of the calculations, the physics should not depend on this scale.

3.1. Vacuum Polarization

The amplitude for the vacuum polarization is given by

$i\mathcal{M}_p =$ 

$$= i\mathcal{M}_0\,\frac{\alpha}{\pi}\left(\frac{1}{3}\log\left(\frac{-q^2}{\mu^2}\right) - \frac{5}{9} + \mathcal{O}(m^2)\right). \tag{4}$$

The differential cross section due to the interference between the leading and the vacuum polarization amplitudes, neglecting the terms that are in $\mathcal{O}(m^2)$, will be

$$\left(\frac{d\sigma}{d\Omega}\right)_{PL} \approx \left(\frac{d\sigma}{d\Omega}\right)_0 \frac{\alpha}{\pi}\left[\frac{2}{3}\log\left(\frac{-q^2}{\mu^2}\right) - \frac{10}{9}\right]. \tag{5}$$

3.2. Electron Self Energy

The amplitude of the electron-self energy in $\overline{\text{MS}}$ renormalization scheme is given by

$$\Sigma_2(\not{p}) = \frac{\alpha}{4\pi}\left[(\not{p} - 2m) + \int_0^1 dx\,(4m - 2x\not{p})\,\log\left(\frac{\mu^2}{(1-x)m^2 - x(1-x)p^2 + xm_\gamma^2}\right)\right], \tag{6}$$

where m_γ is the mass of the photon to regularize the expected IR divergences from the electron self-energy. The Fourier transform of the two point correlation function of the electron self-energy is given by [7]

$$\int d^4x\,\langle\Omega|\,T(\psi(x)\bar{\psi}(0))\,|\Omega\rangle\,e^{ip\cdot x} = \frac{i}{\not{p} - m - \Sigma(\not{p})}. \tag{7}$$

This means that the pole is shifted by $\Sigma(\not{p})$, so the renormalized mass is not the physical mass and the residue of this pole is no longer one [8]. Thus our goal now is to find the correction to the residue and the relation between the renormalized mass m and the physical mass m_e. The physical mass can be given by the position of the pole where we have

$$m_e = m\left[1 + \frac{\alpha}{4\pi}\left(4 + 3\log\left(\frac{\mu^2}{m^2}\right)\right) + O(\alpha^2)\right], \tag{8}$$

while the correction of the residue can be given by the derivative of the electron self-energy amplitude at the physical mass. The residue of the pole will be

$$R = 1 + \frac{\alpha}{4\pi}\left[2\log\left(\frac{m^2}{m_\gamma^2}\right) - \log\left(\frac{\mu^2}{m^2}\right) - 4\right] + \mathcal{O}(\alpha^2). \tag{9}$$

In contrast to the On-shell renormalization scheme, the value of the residue of the pole is no longer one, where the correction to the residue is in $\mathcal{O}(\alpha)$. Which means we have to multiply the amplitude by the value of $R^{1/2}$ for each external leg, which means that we multiply the differential cross section by R^2 [8]. We note that all the corrections will be in higher orders of α except the leading term. So the only affected term by this correction is the leading term, which becomes

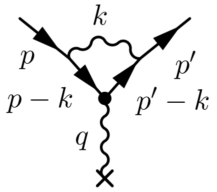
$$\left(\frac{d\sigma}{d\Omega}\right)_L = R^2\left(\frac{d\sigma}{d\Omega}\right)_0 = \left(\frac{d\sigma}{d\Omega}\right)_0\left[1 + \frac{\alpha}{\pi}\left(\log\left(\frac{m^2}{m_\gamma^2}\right) - \frac{1}{2}\log\left(\frac{\mu^2}{m^2}\right) - 2\right)\right] + \mathcal{O}(\alpha^4). \tag{10}$$

The contribution of the self-energy diagram at one loop is zero since the contribution of the diagram due to the self-energy of the incoming electron is exactly the same as the contribution due to the self-energy of the outgoing electron with a relative sign difference.

3.3. Vertex Correction

The amplitude of the vertex correction is given by

$i\mathcal{M}_V =$



$$= \frac{4i\pi\alpha}{q^2}\bar{u}^{s'}(p')\left(\gamma^0\cdot F_1(q^2) + \frac{i\sigma^{0\nu}q_\nu}{2m}F_2(q^2)\right)u^s(p), \tag{11}$$

where $F_1(q^2)$ and $F_2(q^2)$ are the form factors, which in the limit $-q^2 >> m^2$ will be

$$F_1(q^2) \approx \frac{\alpha}{2\pi}\left[-\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{-q^2}{m_\gamma^2}\right) + \frac{1}{2}\log^2\left(\frac{-q^2}{m^2}\right) + 2\log\left(\frac{-q^2}{m^2}\right) - \frac{1}{2}\log\left(\frac{-q^2}{\mu^2}\right) + \frac{\pi^2}{6}\right]. \tag{12}$$

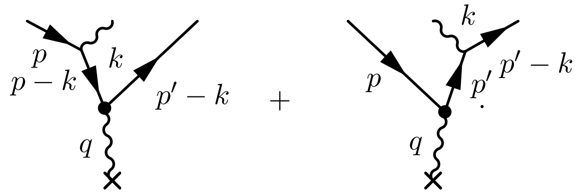
We see that $F_2(q^2)$ is negligible in the limit $m \to 0$. The differential cross section due to the interference between the vertex and the leading amplitudes will be

$$\left(\frac{d\sigma}{d\Omega}\right)_{VL} \approx \left(\frac{d\sigma}{d\Omega}\right)_0 \frac{\alpha}{\pi}\left[-\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{-q^2}{m_\gamma^2}\right) + \frac{1}{2}\log^2\left(\frac{-q^2}{m^2}\right) + 2\log\left(\frac{-q^2}{m^2}\right) - \frac{1}{2}\log\left(\frac{-q^2}{\mu^2}\right) + \frac{\pi^2}{6}\right]. \tag{13}$$

We note that equation (13) contains IR divergences which appear as a single pole when we send m_γ to be zero in addition to the double pole when we set both m_e and m_γ to be zero.

3.4. Bremsstrahlung Correction

The detectors usually can not differentiate between the photon emitted from the vertex and the bremsstrahlung radiation, which require adding the following correction

$i\mathcal{M}_B =$  (14)

Equation (14) represents the diagrams describing the bremsstrahlung correction. According to the Bloch-Nordsieck (BN) theorem, one should sum over all emitted soft photons with energy less than the experimental energy resolution (Δ) to get rid of the IR divergences due to the zero mass of the photon [10]. We consider first the final state soft bremsstrahlung diagrams (i.e an emission of a soft photon either from the incoming and/or the outgoing electrons). In this case, we will be able to use the eikonal approximation which allows us to ignore the linear terms in k from the numerator of the amplitude as $|k| \ll |p' - p|$. The amplitude of the final state soft bremsstrahlung will be

$$i\mathcal{M}_B^{f,S} = ie\,\mathcal{M}_0\left(\frac{p' \cdot \varepsilon^{r*}}{p' \cdot k} - \frac{p \cdot \varepsilon^{r*}}{p \cdot k}\right). \tag{15}$$

The contribution of the final state soft bremsstrahlung to the differential cross section neglecting the terms that are in $\mathcal{O}(m^2, m_\gamma^2)$ will be

$$\left(\frac{d\sigma}{d\Omega}\right)_B^{f,S} \approx \left(\frac{d\sigma}{d\Omega}\right)_0 \frac{\alpha}{\pi}\left[\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{\Delta^2}{m_\gamma^2}\right) - \log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{E^2}{m^2}\right) + \frac{1}{2}\log^2\left(\frac{-q^2}{m^2}\right) + \log\left(\frac{E^2}{m^2}\right) - \log\left(\frac{\Delta^2}{m_\gamma^2}\right) - \frac{\pi^2}{6}\right]. \tag{16}$$

The addition of equations (10), (13) and (16) is free of the IR divergences, but we still have another kind of divergences as we send m to be zero which is called the collinear divergence. Where the detector can not differentiate between an electron and an electron associated with a photon emitted or absorbed collinearly with the incoming or the outgoing electrons. Here we have to use the more general theorem made by Kinoshita, Lee and Nauenberg which is known as the KLN theorem stating that one should sum over both emitted and absorbed hard photons within a cone of an angle less than the experimental angular resolution (δ) [11,12]. The contribution of both initial and final state hard bremsstrahlung will be

$$\left(\frac{d\sigma}{d\Omega}\right)_B^{H} \approx \left(\frac{d\sigma}{d\Omega}\right)_0 \frac{\alpha}{\pi}\left[\log\left(\frac{\delta^2 E^2}{m^2}\right)\left(\log\left(\frac{E^2}{\Delta^2}\right) - \frac{\Delta^2}{2E^2} + \frac{2\Delta}{E} - \frac{3}{2}\right) + \log\left(\frac{\Delta^2}{E^2}\right) - \frac{\pi^2}{3} + \frac{13}{4}\right]. \tag{17}$$

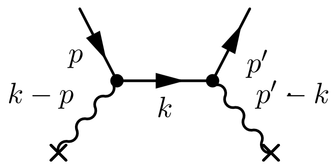
We see that the linear and quadratic terms in Δ can not be ignored as in [12], since it is multiplied by a divergent part producing either a finite piece or divergent piece depending on how small the Δ is. Regarding asking this kind of question we look at similar terms to cancel these suspicious terms. The only way to get similar terms with a relative sign is discussed in [13] by checking the sub-leading collinear divergences from the soft bremsstrahlung which appears beyond the eikonal approximation. Such a calculation requires further work where it should be done carefully as we are interested in the remaining finite pieces from each calculation unlike in [12, 13].

It is obvious that the KLN theorem is the more general form of the BN theorem, but we see that only including the final state soft bremsstrahlung will remove the IR divergences and including both initial and final states hard bremsstrahlung will remove the collinear divergences. Such a treatment with both theorems independently is inconsistent. So we have to include the initial state soft bremsstrahlung which will add more IR divergences that we must take care of. This problem has not been mentioned in [12], while it is been first introduced by [14–16], a more recent discussion can be found in [13]. Where the authors suggest to include the disconnected diagrams to fix the problem. We usually do not add the disconnected diagrams where they describe a non-scattering process. However, the interference between the disconnected diagram with the emission and absorption process produces a fully connected cut diagram. The contribution from adding these diagrams plays an important role in IR cancellation from the initial state as it is shown in [13–16].

Further work needs to be done by including the disconnected diagrams very carefully to get rid of the extra IR divergences from the initial state soft bremsstrahlung and to obtain the remaining finite pieces for the differential cross section.

3.5. Box Correction

The amplitude of the box diagram is given by

$i\mathcal{M}_{BO} =$  (18)

It is obvious that the box diagram does not contain any ultraviolet divergences, so we do not need to perform the dimensional regularization. We use the trick made by R. Dalitz [17] to simplify the integrals in this diagram. The differential cross section due to the interference between the leading and the box amplitudes will be

$$\left(\frac{d\sigma}{d\Omega}\right)_{BOL} = \frac{\pi\alpha^3 E}{p\,Q\,q^2}\,(p-Q) + O(\alpha^4)\,,\ Q = |q|. \tag{19}$$

3.6. NLO correction to the differential cross section in Rutherford Scattering

Now we include the NLO contributions mentioned above to the differential cross section without including the initial state soft bremsstrahlung and the sub-leading terms from the soft bremsstrahlung beyond the eikonal approximation (as discussed above, they require more careful work). Since μ is arbitrary, We choose it to be $-q^2$, then we find

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_0 \Bigg[1 + \frac{\alpha}{\pi}\Bigg(\log\left(\frac{\Delta^2}{E^2}\right)\left(1 + \log\left(\frac{-q^2}{E^2\delta^2}\right)\right) + \frac{3}{2}\log\left(\frac{-q^2}{E^2\delta^2}\right) - \frac{\pi^2}{3} + \frac{5}{36} + \log\left(\frac{\delta^2 E^2}{m^2}\right)\left(\frac{2\Delta}{E} - \frac{\Delta^2}{2E^2}\right)\Bigg)\Bigg] + \frac{\pi\alpha^3 E}{p\,Q\,q^2}\,(p-Q) + O(\alpha^4). \tag{20}$$

4. Conclusion

In this paper, we calculated the elastic scattering differential cross section, including the next-to-leading order corrections of a massless electron scattered by a classical static point charge. These corrections come from the photon self-energy, vertex, bremsstrahlung and the box diagrams. We found that all UV divergences are absorbed by the counter terms in the $\overline{\text{MS}}$ renormalization scheme. We also saw that unlike most of the renormalization schemes, using $\overline{\text{MS}}$ shows that the contribution from the vacuum polarization correction remains finite in the zero mass limit.

Applying the BN theorem provides the usual cancellation of the IR divergences from the vertex with the one from the final state soft bremsstrahlung. We also note that applying the KLN theorem removes all the collinear divergences by including both initial and final state hard bremsstrahlung. But we still need to add the initial state soft bremsstrahlung correction to stay in the spirit of the more general KLN theorem.

More work to be done by checking the calculations of the soft bremsstrahlung beyond the eikonal approximation as well as including the disconnected diagrams that contribute with the same order of α to get a finite form of the differential cross section; equivalently, we expect a result that is valid up to arbitrary large momentum exchange. Our result also satisfies the Callan-Symanzik equation [18], where it is straightforward to check that the differential cross section at NLO is independent on the mass scale μ.

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