Thomas Rotation and Quantum Entanglement.

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Abstract. The composition of two non-linear boosts on a particle in Minkowski space-time are not commutative. This non-commutativity has the result that the Lorentz transformation formed from the composition is not a pure boost but rather, a combination of a boost and a rotation. The rotation in this Lorentz transformation is called the Wigner rotation. When there are changes in velocity, as in an acceleration, then the Wigner rotations due to these changes add up to form Thomas precession. In curved space-time, the Thomas precession combines with a geometric effect caused by the gravitationally curved space-time to produce a geodetic effect. In this work we present how the Thomas precession affects the correlation between the spins of entangled particles and propose a way to detect forces acting on entangled particles by looking at how the Thomas precession degrades the entanglement correlation. Since the Thomas precession is a purely kinematical effect, it could potentially be used to detect any kind of force, including gravity (in the Newtonian or weak field limit). We present the results that we have so far.

1. Introduction

While Bell's theorem is well known and was originally presented in John Bell's 1964 paper as a way to empirically test the EPR paradox, this was only for non-relativistic quantum mechanics. It was only fairly recently that people have started investigating possible relativistic effects on entanglement and EPR correlations beginning with Czachor [1] in 1997. Czacor found that relativistic effects of massive particles effect the correlation between entangled massive particles and depends on the velocity of the particles in the simple case where both particles are moving in the same direction. The origin of this apparent deviation from the classical violation of Bell's inequality appears to be a length contraction when measuring the spin correlations in the laboratory frame.

Later authors then also started revisiting this issue. In 2003, Terashima and Ueda [2] showed that an observer moving perpendicular to the motion of the entangled particles with opposite spins, moving in opposite directions, will observe a degradation of the entanglement correlation when measured in the usual directions as opposed to an observer that's in the frame of the source. However, they concluded that the entanglement and quantum information is still preserved using the reasoning that one can still get the maximal violation of Bell's inequality if the spins are measured in compensating directions, so the situation is still non-local. The apparent decoherence effect found by Terashima and Ueda is due to a Wigner rotation as measured in the moving frame. It is interesting that they also calculated such an effect for massless entangled particles moving at the speed light c. In 2004, Lee and Chang-Young [3] calculated the general case, combining Czachor's result with that of Terashima and Ueda and derived the directions in which maximal violation of Bell's inequality occurs in vector form.

Some authors also investigated quantum communication in accelerated frames [4] [5]. However, these authors didn't investigate quantum correlations but rather specific forms of quantum communication such as teleportation. There have also even been recent works investigating how curved space-time affects quantum correlations [6] [7].

2. Maximal violation of Bell's inequality of entangled massive relativistic particles.

In order to understand the Wigner rotation and how it affects the correlations between entangled relativistic particles, let's consider the setup as described by Terashima and Ueda [2]. In their setup, they considered 2 particles moving in opposite directions from a source in the x and $-x$ directions respectively and the 2 observers, Alice and Bob, moving at the same velocity, perpendicular to the direction of motion of the 2 particles. A Wigner rotation is a spacial rotation in the coordinates between 2 reference frames when the transformation to the new frame is the result of a succession of 2 non-linear Lorentz boosts. In this case, the first boost is a Lorentz boost from the rest frame of each particle respectively to the rest frame of source, where the particle is moving at a constant velocity $\vec{v}$. The second boost, is a boost from that frame, perpendicular to the motion of the particles, to get to the observer frame of Alice and Bob. Terashima and Ueda found that if, in the rest frame of the source, the 2 particles are in the singlet state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(\left|\vec{p}_+, \frac{1}{2}\right\rangle \left|\vec{p}_-, -\frac{1}{2}\right\rangle - \left|\vec{p}_+, -\frac{1}{2}\right\rangle \left|\vec{p}_-, \frac{1}{2}\right\rangle\right), \tag{1}$$

then applying the unitary operator $U(\Lambda)$ corresponding to the second boost $\Lambda^{\mu}{}_{\nu}$, they get

$$\begin{aligned} U(\Lambda)\,|\Psi\rangle &= \frac{1}{\sqrt{2}}\left[\cos\delta\left(\left|\vec{\Lambda p}_+, \frac{1}{2}\right\rangle \left|\vec{\Lambda p}_-, -\frac{1}{2}\right\rangle - \left|\vec{\Lambda p}_+, -\frac{1}{2}\right\rangle \left|\vec{\Lambda p}_-, \frac{1}{2}\right\rangle\right)\right. \\ &+ \left.\sin\delta\left(\left|\vec{\Lambda p}_+, \frac{1}{2}\right\rangle \left|\vec{\Lambda p}_-, -\frac{1}{2}\right\rangle + \left|\vec{\Lambda p}_+, -\frac{1}{2}\right\rangle \left|\vec{\Lambda p}_-, \frac{1}{2}\right\rangle\right)\right], \end{aligned} \tag{2}$$

where Λp represents the momenta as measured by Alice and Bob and δ is the Wigner angle given by

$$\tan\delta = \frac{\sinh\xi\sinh\chi}{\cosh\xi + \cosh\chi}, \tag{3}$$

where ξ is the rapidity of the rapidity of the particles in the rest frame of the source, whose velocity is given by $\frac{v}{c} = \tanh\xi$ and χ is the rapidity of the observer frame (from which Alice and Bob measure the 2 particles) with respect to the rest frame of the source, with it's relative velocity given similarly as $\frac{V}{c} = \tanh\chi$. In order to find out how this affects Bell's theorem, Terashima and Ueda made use of the CHSH observable

$$C(a, a', b, b') \equiv \left\langle \hat{a}\otimes\hat{b}\right\rangle + \left\langle \hat{a}\otimes\hat{b'}\right\rangle + \left\langle \hat{a'}\otimes\hat{b}\right\rangle - \left\langle \hat{a'}\otimes\hat{b'}\right\rangle, \tag{4}$$

where the CHSH-inequality is given classically by $C(a, a', b, b') \leqslant 2$ but quantum mechanics predicts this observable to have a maximum value of $C(a, a', b, b') = 2\sqrt{2}$. Terashima and Ueda showed that

$$C(a, a', b, b') = 2\sqrt{2}\cos^2\delta, \tag{5}$$

in their setup if the vector set is chosen such that the observable would have the maximum value $2\sqrt{2}$ in the non-relativistic case and thus the inequality may not be violated depending on the relative velocities between the 2 frames and the boosts. They did reason, however, that because the difference is by an angle, a different vector set can be chosen so that the maximum violation of Bell's inequality can still be recovered even in their setup as one merely has to rotate the original vectors though the Wigner angle δ.

Terashima and Ueda however, did not define a relativistic spin observable. One of the very first papers on the EPR paradox for relativistic particles did though. This is what Czachor did in a paper published in 1997 [1]. He considered an even simpler setup to Terashima and Ueda though, where both entangled particles were moving in the same direction. Using his relativistic spin observable, he showed some kind of degradation in the correlation even occurs not only when the particles are moving parallel to each other but also when they're moving in the same direction. In this case though, the difference is due to Lorentz contraction in the direction of motion and is not a Wigner angle. Czachor concluded that when the entangled particles are relativistic, then the Bell inequalities may not be violated. However, later on Lee and Chang-Young showed that in this case too, the maximal violation can again be recovered, simply by choosing a different vector set [3]. They also added this difference as a correction on Terashima and Ueda's result to get a more complicated formula

$$\begin{aligned} C(a,a',b,b') &= \frac{2}{\sqrt{1+\sin^2\theta_\Lambda+\cosh^2\eta\cos^2\theta_\Lambda}} \\ &+ \frac{2}{\sqrt{1+\sin^2\theta_\Lambda+\cosh^2\eta\cos^2\theta_\Lambda}\sqrt{\sin^2\theta_\Lambda+\cosh^2\eta\cos^2\theta_\Lambda}} \\ &\times \{[(\cosh\eta\cos^2\theta_\Lambda-\sin^2\theta_\Lambda)^2-(1+\cosh\eta)^2\sin^2\theta_\Lambda\cos^2\theta_\Lambda]\cos 2\delta \\ &- (1+\cosh\eta)(\cosh\eta\cos^2\theta_\Lambda-\sin^2\theta_\Lambda)\sin 2\theta_\Lambda\sin 2\delta\}, \end{aligned} \tag{6}$$

where the angle θ_Λ is the rotation to the direction of the momentum $\vec{\Lambda p}$ and $\tanh\eta = \frac{\sqrt{(\tanh^2\xi+\sinh^2\chi)}}{\cosh\chi}$.

3. Corrected Bell observables for massive relativistic particles

In the setup considered by Terashima and Ueda, one only has to rotate the vectors in the vector set that give the maximal violation of Bell's inequality by the Wigner angle δ in order to get the corrected vector set that regains the maximal violation. Lee and Chang-Young, however, derived the following components of the corrected vectors $\vec{a}_c$ and $\vec{b}_c$ as used in the joint-spin measurement in their paper,

$$a_{cz} = \frac{\bar{a}_z}{\sqrt{[F_a(1+\cosh\eta)\sin\theta_\Lambda\cos\theta_\Lambda-(\sin^2\theta_\Lambda-\cosh\eta\cos^2\theta_\Lambda)]^2-\bar{a}_z^2\sinh^2\eta(F_a\sin\theta_\Lambda+\cos\theta_\Lambda)^2}}, \tag{7}$$

$$a_{cx} = \bar{a}_x\sqrt{1+a_{cz}^2\sinh^2\eta(F_a\sin\theta_\Lambda+\cos\theta_\Lambda)^2}, \tag{8}$$

$$a_{cy} = a_y\sqrt{1+a_{cz}^2\sinh^2\eta(F_a\sin\theta_\Lambda+\cos\theta_\Lambda)^2}, \tag{9}$$

$$b_{cz} = \frac{\bar{b}_z}{\sqrt{[F_b(1+\cosh\eta)\sin\theta_\Lambda\cos\theta_\Lambda - (\sin^2\theta_\Lambda - \cosh\eta\cos^2\theta_\Lambda)]^2 - \bar{b}_z^2\sinh^2\eta(F_b\sin\theta_\Lambda - \cos\theta_\Lambda)^2}}, \tag{10}$$

$$b_{cx} = b_x\sqrt{1 + b_{cz}^2\sinh^2\eta(F_b\sin\theta_\Lambda - \cos\theta_\Lambda)^2}, \tag{11}$$

and

$$b_{cy} = b_y\sqrt{1 + b_{cz}^2\sinh^2\eta(F_b\sin\theta_\Lambda - \cos\theta_\Lambda)^2}, \tag{12}$$

where

$$F_a = \frac{(1+\cosh\eta)\tan\theta_\Lambda - f_a(\tan^2\theta_\Lambda - \cosh\eta)}{(1+\cosh\eta\tan^2\theta_\Lambda) - f_a(1+\cosh\eta)\tan\theta_\Lambda}, \tag{13}$$

and

$$F_b = -\frac{(1+\cosh\eta)\tan\theta_\Lambda + f_b(\tan^2\theta_\Lambda - \cosh\eta)}{(1-\cosh\eta\tan^2\theta_\Lambda) + f_b(1+\cosh\eta)\tan\theta_\Lambda}, \tag{14}$$

where $f_a \equiv \frac{\bar{a}_x}{\bar{a}_z}$ and $f_b \equiv \frac{\bar{b}_x}{\bar{b}_z}$.

4. Addition of accelerations to the Wigner angle and Thomas precession

As shown by Czachor, the vector basis required for the maximal violation of Bell inequalities does change even when both entangled particles are moving in the same direction. However, this difference can be made to be zero if the spin-directions are measured perpendicularly to the direction of motion. So, for now at least, in our description of including acceleration, we will simply impose an acceleration on Terashima and Ueda's setup. An acceleration can be thought of as successive additions of small velocities. So in Terashima and Ueda's setup, let's consider the particles to be accelerating in their direction of motion. This acceleration could then be thought of as many additions of velocities. So using the same symbols as in previous sections, if we take $\vec{v}$ and $-\vec{v}$ as the initial velocities of the particles in opposite directions, as measured in the rest frame of the source and assume that they're also accelerating in opposite directions then this can be expressed as

$$\frac{v \oplus \Delta v}{c} = \tanh(\xi + \Delta\xi), \tag{15}$$

where $\oplus$ denotes the addition of velocities using the relativistic addition of velocities formula and the value for the other particle is just minus the above formula. The total change in velocity Δv is given by $\Delta v = \bigoplus_i^N \Delta a_i \Delta t$ where each $\frac{\Delta a_i \Delta t}{c} = \tanh(\Delta a_{Pi}\Delta\tau)$. Here Δa_i are the relativistic accelerations and Δa_{Pi} are the proper accelerations respectively corresponding to the relativistic time Δt as measured in the rest frame of the source and the proper time $\Delta\tau$ as measured in the rest frame of the particles. This gives $\frac{\Delta v}{c} = \tanh\left(\sum_{i=1}^N \Delta a_{Pi}\Delta\tau\right)$ and $\Delta\xi = \sum_{i=1}^N \Delta a_{Pi}\Delta\tau$. As $N \to \infty$, $\Delta\tau \to 0$, therefore $\lim_{N\to\infty}\Delta\xi = \int_0^\tau a_P(\tau)d\tau$.

The reason for choosing the acceleration as being in the same or opposite to the direction of motion of the particles is because the addition of all these infinitesimal velocities, and thus the acceleration, would not change their direction of motion as measured in the rest frame of the source, which is why equation (15) is valid. Thus this is a simple case. However, when applying the second boost to the observer frame from where Alice and Bob do their measurement of the particles, the acceleration does change the direction of motion as measured in this frame, where the change in Wigner angle is given by

$$\tan(\delta_0 + \Delta\delta) = \frac{\sinh(\xi + \Delta\xi)\sinh\chi}{\cosh(\xi + \Delta\xi) + \cosh\chi}. \tag{16}$$

Each velocity gained or lost over every infinitesimal unit of time adds a little bit or takes away a little bit from the Wigner angle such that a precession is observed. This is the Thomas precession and is also responsible for the spin-orbit coupling of an electron in a hydrogen atom. However, it has only recently been realised, since the first paper on the subject in the '90's, that Thomas precession also affects correlations between entangled relativistic particles. It was probably known known before but it was since then that the subject started to be studied in more detail. Others have also studied quantum communication in accelerated frames but those papers were studying the effects on specific phenomena like quantum teleportation, not the pure effects of acceleration on it's own. Our motivation for doing this is the possibility of using such differences to measure forces between particles and investigating whether such an approach would be more sensitive than measuring them directly, such as looking for a loss in energy. If it were indeed a more sensitive way to measure and observe forces, then it may be possible to exploit the entanglement between particles in order to detect forces that would otherwise be too small to measure.

One example that we did think of is the possibility of directly measuring the gravitational force on the femto-metre scale in the Newtonian approximation. We did this by developing a very simple model of how a Newtonian gravitational force of a decaying heavy particle would affect the daughter particles. Due to the weakness of gravity, one might think that the gravitational force of elementary particles would be extremely tiny. We found however, that on the scales we're looking, the gravitational acceleration of the elementary particles are surprisingly large and this is due to the inverse square law. Unfortunately, however, changes in velocity due to the gravitational force are still extremely tiny (on the order of about 10^{-25}) and though the entanglement correlations were found to be more sensitive than measuring such changes directly in some cases, this higher sensitivity may still not be enough to detect something as weak as that. This is an extreme case though and we can still imagine some more practical applications we weak forces may be able to be detected. We've only so far looked at Bell observables and not tried applications of optimal entanglement witnesses which also get used by the method of quantum state tomography, which are likely to be an approach which is more sensitive anyway. We will not give the full derivation of this gravitational model here as it is being prepared for publishing in a larger journal. Let's instead consider the much simpler case of a uniform acceleration. In this case $\frac{\Delta v}{c} = \tanh(a_P\tau)$ and therefore, equation (16) will be given as

$$\tan(\delta_0 + \Delta\delta) = \frac{\sinh(\xi + a_P\tau)\sinh\chi}{\cosh(\xi + a_P\tau) + \cosh\chi}, \tag{17}$$

where a_P is the proper acceleration and τ is the total proper time. As you can see, if you can measure the total change in Wigner rotation $\Delta\delta$, then in principle you can calculate the proper acceleration a_P and therefore measure the forces acting on the particles. The same is true of non-uniform accelerations as in the case of the gravitational force that we spoke of above. In principle, all one has to do to measure the difference in the Wigner rotation is to use the Bell observable, just measure the angle by which one has to rotate the detector in order to get the maximal violation, so this difference is directly measurable in principle.

5. Conclusion

In this paper we have given a neat summary of previous work on quantum entanglement between relativistic particles and their correlations. Czachor found that Lorentz contraction affects the angle of the spins of the 2 entangled particles as measured by moving observers even when the particles are moving in the same direction. Terashima and Ueda found that the Wigner angle resulting from a succession of 2 boosts gives rise to a measurable difference in entanglement correlations. Lee and Chang-Young combined the results of these previous papers and gave their own derivation of the corrected vector set required to recover the maximal violation of Bell's inequalities. Finally we added our own input by adding an acceleration to Terashima and Ueda's setup and discussed some possible applications of this. We conclude that it may be possible to use these differences in entanglement correlations to detect forces between particles but this is not certain yet. More investigation with regards to this point is required.

References

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