Non-universality in a constrained period doubling route to chaos for Rössler's system

André E Botha[1] **and Wynand Dednam**[1,2]

[1]Department of Physics, University of South Africa, Science Campus, Private Bag X6, Florida 1710, South Africa

[2]Departamento de Fisica Aplicada Universidad de Alicante, San Vicente del Raspeig, E-03690 Alicante, Spain

E-mail: bothaae@unisa.ac.za

Abstract. Some important questions concerning the existence of periodic orbits in chaotic systems were investigated for the 3-dimensional Rössler system at different parameter values. While previous studies have classified the periodic orbits by varying only one or two of the three parameters at a time, the present study made use of optimization to find a continuous curve in parameter space corresponding to periodic orbits through one specific point in state space. The set of periodic orbits through the specific point underwent an interesting series of bifurcations that was different to the unconstrained canonical route described by Feigenbaum. It was concluded that the idea of classifying periodic orbits by grouping them into sets that pass through specific points of state space may provide further insight into clustering of orbits and the apparent relationship between the point dimension in parameter space and the Lyapunov dimension of the chaotic attractor at the corresponding parameters. Constrained stable periodic orbits may also have several practical applications, such as electrical timing circuits.

1. Introduction

State spaces of systems exhibiting low-dimensional chaos are replete with unstable periodic orbits that exist in the midst of chaotic attractors. Ever since this realization, which only took place in the 1990s, attempts have been made to characterize chaotic attractors via the properties (such as the characteristic exponents) of unstable periodic orbits [1]. In addition to being of a fundamental importance to understanding of chaotic dynamics, the properties of periodic orbits in chaotic systems is also of practical importance, since several methods have been devised by which periodic orbits may be extracted directly from experimentally measured time series [2]. The detection of periodic orbits in time series data is a good test for determinism and is also a starting point for many current methods of chaos control [3].

Recently Botha and Dednam [4] conjectured that periodic orbits exist through any point in the state space of the well-known 3-dimensional Rössler system [5]. The validity of their conjecture was demonstrated by using an optimized shooting method [6] to find periodic orbits corresponding to 10 000 randomly chosen initial conditions (x_0', y_0', z_0'), where x_0', y_0', $z_0' \in [-100, 100]$. (See Ref. [4] for details.) For all the tested initial conditions, the optimized shooting method was able to determine at least one periodic orbit through each initial condition, without any exceptions. Furthermore, it was noted that two or more distinct sets of the optimized parameters (giving periodic orbits) could be found for the same initial condition. In the present article, the consequences of this multiplicity of parameter sets corresponding to a periodic orbit through one particular point will be explored. For simplicity, in our subsequent discussions, we will refer to all periodic solutions that pass through the same point, simply as a *state*.

One may thus ask, how many sets of parameters (typically) can be found for a given state? Could it perhaps be infinite? To answer this question, we consider the case in which a periodic solution is structurally stable within a given range of parameters. If one changes two of the parameters, say a and b in Rössler's system, the periodic solution should sweep its neighborhood two-dimensionally, unless the direction of sweep is degenerate. Therefore, in general, the three-dimensional space near the periodic solution may be expected to be fully covered by changing only two of the system parameters. For this reason, one may at first think that an infinite set of parameters, for which a periodic solution passes the given state, may exist in a one-parameter family. However, this is not the case. In general, the periodic orbit undergoes a series of saddle-node bifurcations, which follow the well-known Feigenbaum scenario [7, 8], i.e. as one of the parameters is varied, there is a cascade of periodic doubling orbits, eventually leading to fully developed chaos. As an example, consider the bifurcation of a period-(n-1) solution into a period-n solution. Initially the period-(n-1) solution is stable. As the single parameter is varied, it becomes progressively less stable in one direction only, until it becomes unstable, at which point a period-n solution is born. Immediately after this saddle-node bifurcation, the trajectory of the period-n orbit can be thought of as lying on the edge of a Möbius strip containing the unstable period-(n-1) orbit [9]. The unstable manifold of the period-(n-1) orbit lies along the surface of the Möbius strip. Thus in the canonical Feigenbaum scenario, period-n solutions are created by repulsion away from period-(n-1) solutions. One would therefore expect a period-n orbit to have no points in common with the period-(n-1) orbit from which it originated. When only one parameter is varied, this is indeed the case. However, if one considers bifurcations that occur when more than one parameter is allowed to vary at a time, then it is possible to obtain bifurcated orbits with one or more points in common.

2. Model and computational techniques

For the purposes of doing numerical calculations it is useful to write Rössler's system in terms of scaled coordinates. First, let $T > 0$ be some characteristic time in the system and rescale the time t' in terms of an unprimed (dimensionless) coordinate t, where $t' = Tt$. Here T could be the period of a periodic solution or, in the case of a chaotic attractor, an average orbital time. Second, choose a dimensionless scaling factor $\alpha \geq 1$, such that $x' = \alpha x$, $y' = \alpha y$, $z' = \alpha z$. In these unprimed coordinates, the Rössler system is given by

$$\dot{x} = T\left(-y - z\right), \quad \dot{y} = T\left(x + ay\right), \quad \dot{z} = T\left(b/\alpha + \alpha xz - cz\right). \tag{1}$$

The form of (1) has several advantages. First, the scaled coordinates x, y, and z never become too large, since one can always increase the value of α to ensure that they remain bounded below a certain threshold. This boundedness of the coordinates facilitates more accurate numerical integration at large values of the primed coordinates. Second, (1) can be integrated conveniently over exactly one period T by integrating t over the closed interval $[0, 1]$. Moreover, the integration can be done without *a priori* knowledge of T, which is determined later, together with the other parameters, via Levenberg-Marquardt optimization.

In the present work we have modified the integration scheme that was previously used in our optimized shooting method. Instead of the 5th order (fixed time step) Runga-Kutta method which we used previously, we now employ the Hairer and Wanner implementation of the Dormand-Prince embedded method of order (4)5 [10]. For the same global accuracy, the latter method is significantly faster because it employs adaptive time stepping. It also has the advantage of allowing control over the local relative and absolute errors with dense output.

3. Results

The idea of clustering of points in the parameter space, corresponding to periodic orbits of (1), has been considered previously [11–16]. These works considered different methods for generating peculiar "swallow" or "shrimp" shaped regions of periodic orbits embedded in chaotic regions in two-parameter space, usually by fixing one of the parameters and then varying the other two. The "swallows" or "shrimps" then emerged when the maximum Lyapunov exponent was plotted as a function of the varying parameters. The "swallow" or "shrimp" shaped regions appeared as universal features of certain continuous chaotic flows [11, 15, 16].

In the present work we establish the distribution of periodic orbits as functions of the parameters. We therefore consider a random set of initial conditions which is constructed to have a uniform average density of 100 points per unit volume of state space, i.e. although the points are randomly chosen, they are chosen so that the point dimension (See equation 11 of Ref. [2].) of any initial condition is close to 3, which is the dimension of the state space. We then find the set of parameters corresponding to all possible periodic orbits through the chosen set of initial conditions. These periodic orbits were found by simultaneously optimizing the three control parameters and the period. In figures 1(a) and (b) we show two-dimensional projections of the parameter space of periodic points. In this case we considered 10800 initial conditions (in the positive half space $z_0 > 0$) of a $6 \times 6 \times 6$ cube centered on the origin. Only orbits with $1 < T < 50$ were sought.

As can be seen in figure 1, the periodic points generally appear to be most densely packed nearer to the origin, where the system in fact becomes integrable [17]. However, by calculating the point dimension of a selection of points we have shown that the distribution is non-uniform and that the point dimension of the parameter space is quite close to the Lyapunov dimension of Rössler's attractor at typical parameter values (See, for example, Ref. [18].) This result may just be coincidental, but it does warrant further investigation. At present we cannot postulate a reason why the point dimension in parameter space is close to the Lyapunov dimension of the attractor.

We conclude this section by noting that the found parameter points were all consistent with previous theoretical considerations that preclude the existence of periodic orbits in certain regions of the parameter space [13]. For example, in the present case, periodic orbits may only exist for $c^2 - 4ab \geq 0$, and with either (i) all parameters having the same sign (as in figure 1), (ii) $a < 0$, $b < 0$, $c > 0$ and $ab/c \geq c - 1/a - a$ (not shown), (iii) $a > 0$, $b > 0$, $c < 0$ and $ab/c \leq c - 1/a - a$ (not shown).

As explained in the introduction, the bifurcations that occur on orbits passing through the same point are fundamentally different from standard period-doubling bifurcations. To obtain such a sequence of bifurcations we arbitrarily chose some initial condition $\boldsymbol{x}_0$ and repeatedly searched for periodic orbits through $\boldsymbol{x}_0$, by using our optimized shooting method to optimize the parameters a, b and period T . The third system parameter was not part of the optimization. After obtaining the first periodic orbit corresponding to (a_1, b_1, c_1, T_1), we increased the value of c by an amount $\Delta c = 0.001$ and used the parameter values $a_1 \pm r_{11}, b_1 \pm r_{12}$ and period $T_1 \pm r_{13}$ as the initial guess for searching for the next periodic orbit. Here the r_{1j} $(j = 1, 2, 3)$ were random numbers ranging from just under ten percent to just over ten percent of previously found values of a, b and T. For example, after finding a_1, the initial guess for the next a was randomly chosen from the open interval $(a_1 - a_1/10, a_1 + a_1/10)$. Similarly for b and T. This procedure was repeated for $c \in [1, 5]$ and it produced about 10 000 orbits. As the value of c was systematically increased, the method sometimes failed to find a periodic orbit without exceeding the numerical tolerance of the residual, which was required to be less than 10^{-12} in this calculation. To proceed, we then guessed a period of twice the maximum value that was obtained at the last successful value of c. With the initial guess for the period doubled, the method was then once again able to find new periodic orbits and the values for c could once

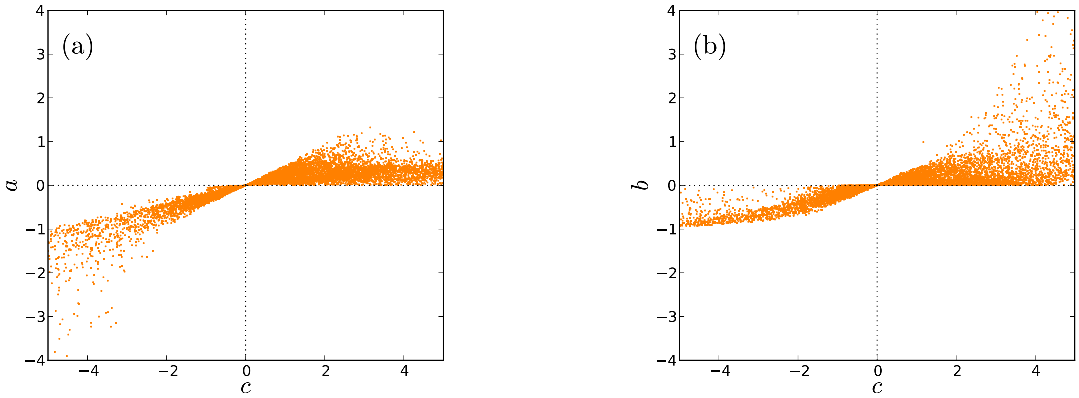

Figure 1. Clustering of points in parameter space, corresponding to periodic orbits of a uniformly dense set of initial conditions. (a) the ac-projection and (b) the bc-projection of the parameter space. Each figure contains 32243 points, with the highest density of points occurring closer to the origin, as discussed in the main text.

again be increased systematically. To ensure that all bifurcations were found, we repeated this procedure several times for systematically increasing and decreasing values of c. In this way we obtained a complete set of periodic orbits (with $1 < T < 50$) passing through $\boldsymbol{x}_0$.

In the remainder of this section we will discuss the periodic orbits through one particular initial condition, which we have randomly chosen to be $\boldsymbol{x}_0 = (1.483, -5.182, 0.0548)$. On the left hand axis of figure 2 are shown the parameters a and b as functions of c, as well as the stability μ as a function of all three parameters. More precisely, μ is the modulus of the maximum non-trivial Floquet multiplier, which has been calculated here according to the method developed by Lust [19]. Accordingly, a periodic solution is asymptotically stable if the modulus of each Floquet multiplier, except the trivial one, is strictly less than one. Otherwise, if one or more of the multipliers is greater than one in modulus, the solution is asymptotically unstable. In figure 2 we have plotted the unstable orbits as dashed lines. The scale on the right hand axis gives the periods of the orbits, which are plotted as a thicker green solid or dashed line. For clarity of presentation we have not included all sequences of periodic solutions that were found.

One interesting new feature of figure 2 is the appearance of unstable periodic orbits, apparently out of nowhere. Moreover, at certain values of c (for example, $c = 2.921$, 3.877 and 4.117) one can see that stable orbits become neutrally stable ($\mu = 1$). At such points there are subharmonic bifurcations from unstable to stable (or vice versa) orbits. Moreover, by looking at the full spectrum of Floquet multipliers immediately before and after the bifurcation, we observe that the bifurcation is not of the usual saddle-node type. Thus the period doubling bifurcations that occur in situations where one parameter is varied without any constraints, do not appear to be *generic*, i.e. their basic character is here seen to be altered by the perturbations that are necessary to maintain the orbit precisely through one point. This is an interesting new aspect of the bifurcations found here.

Lastly, we would like to emphasize that not all of the transitions that can be seen in figure 2, from one type of periodicity to another, are in fact bifurcations. In some cases (most notably at $c \approx 2.2$ and 3.75) the transitions occur for discontinuous values of a and/or b. When viewing figure 2, one thus has to bear in mind that, by definition, bifurcations only occur for infinitesimal change in one or more of the parameters. The inset to figure 2 also reminds us that the unstable orbits may in general lose stability in a variety of ways. In this bifurcation diagram, which

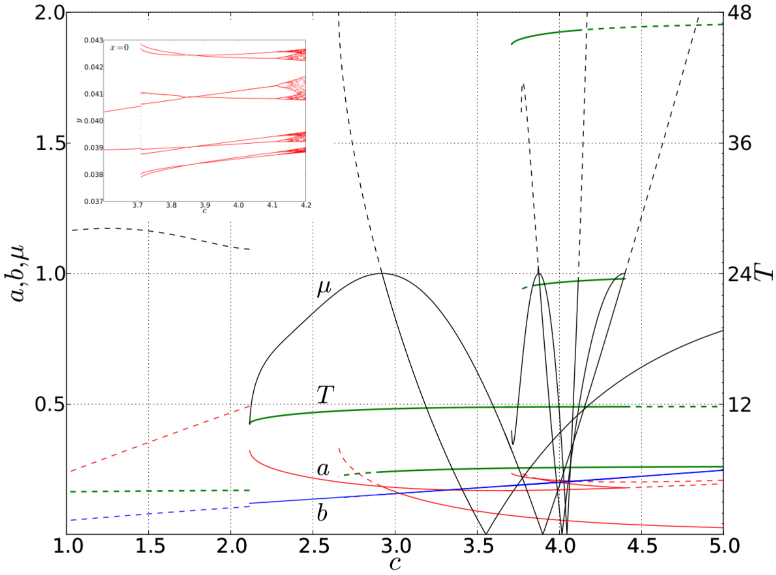

Figure 2. Sequences of periodic solutions through the arbitrary initial condition $\boldsymbol{x}_0 = (1.483, -5.182, 0.0548)$. The left hand axis shows the parameters a (red) and b (blue) as a function of the third control parameter c. Also plotted as a function of c is the largest Floquet multiplier μ (black). The right hand axis shows the period T (green) of the periodic orbit, also as a function of c. Solid lines correspond to stable periodic orbits, while the unstable orbits are indicated by dashed lines. The inset shows the bifurcation diagram obtained from the Poincaré section $x = 0$ for the orbit with $T \approx 47$ and over the range $c \in [3.6, 4.2]$. See text for details.

is a Poincaré section on the plane $x = 0$, we see that the initially stable period-8 orbit (with $T \approx 47$), first decays to a stable period-16 orbits (if they were not stable it would not appear on the bifurcation diagram) and then into chaotic attractors (which can be see to the far right of the diagram), as c is increased from roughly 4.1 to 4.2. Also, it is obvious that traditional bifurcation diagrams, such as the one in the inset, would not obey Feigenbaums universal scaling laws, simply because these law apply only to situations in which one parameter is varied at a time. In the present situation, all three parameters are varied at once.

4. Conclusion

The constrained period doubling route to chaos which we have described for Rössler's system is relevant to a wide class of systems, the dynamics of which can all be described in terms of an underlying unimodal map [1]. Here we have suggested an alternative way of looking at periodic orbits. Instead of fixing one of the system parameters and then considering the remaining two-dimensional parameter space, we have considered a specific point in the state space and varied all three parameters to obtain all periodic orbits passing through the specific point. Our results demonstrate that the periodic orbits correspond to curves in the parameter space, which are mostly smooth curves, i.e. there are only a small number of discontinuities. This is an unexpected result. Furthermore, our numerical investigations have revealed that period doubling bifurcations may not be generic, as was previously thought. We have obtained the curious result that the point dimension of points in the parameter space, corresponding to a uniform distribution of initial conditions in the state space, appears to be almost the same as the Lyapunov dimension of the attractor at typical parameter values.

Other than being of mathematical interest, our results may also have some practical applications. There are in fact many practical situations where one may be interested in imposing constraints on a system. For example, the Rössler system itself has been implemented in electrical circuits for use in encryption, studies of synchronization, and of course chaos control [20]. These days systems of nonlinear equations are routinely implemented (mainly for use in chaos encryption) in field programmable gate arrays (FPGAs), which offer several advantages over analog electrical circuits. (See, for example, Ref. [21].) There is thus a real possibility of accurately varying the parameters of a system by changing, for example, resistances and capacitances of the constituent electrical components. As our results demonstrate, a careful choice of the system parameters may be used to engineer stable periodic signals of essentially any periodicity, while always retaining a specific value of the periodic signal. Such signals are important for making timing devices, where very specific values of a periodic signal may be required to trigger certain parts of more complicated electrical or FPGA circuits.

Acknowledgments

AEB would like to thank Craig Thompson for his informal participation in this research. WD thanks the National Research Foundation of South Africa for financial assistance in the form of a Scarce Skills Masters Scholarship (Grant Unique Number 92138).

References

[1] Hilborn R C 2000 *Chaos and Nonlinear Dynamics: An Introduction* 2nd ed (New York: Oxford University Press)

[2] So P, Ott E, Sauer T *et al.* 1997 *Phys. Rev. E* **55** 5398

[3] Ott E and Spano M 1995 *Physics Today* **48** 34

[4] Botha A E and Dednam W 2014 *Proceedings of the 59th Annual Conference of the SAIP* ed Karataglidis S and Engelbrecht C (7-11 July, University of Johannesburg)

[5] Rössler O E 1976 *Phys. Lett. A* **57** 397

[6] Dednam W and Botha A E 2015 *Engineering with Computers* **31** 749

[7] Feigenbaum M J 1978 *Journal of Statistical Physics* **19** 25

[8] Feigenbaum M J 1979 *Journal of Statistical Physics* **21** 669

[9] Ott E 1993 *Chaos in Dynamical Systems* (Toronto: Cambridge University Press)

[10] Hairer E, Nørsett S P and Wanner G 2008 *Solving Ordinary Differential Equations I: Nonstiff Problems* (Berlin: Springer-Verlag)

[11] Gomez F, Stoop R L and Stoop R 2014 *Bioinformatics* **30** 2486

[12] Castro V *et al.* 2007 *Int. J. Bifurcation Chaos* **17** 965

[13] Starkov K E and Starkov K K 2007 *Chaos, Solitons and Fractals* **33** 1445

[14] Gallas J A C 2010 *Int. J. Bifurcation Chaos* **20** 197

[15] Zou Y, Donner R V, Donges J, Marwan N and Kurths J 2010 *Chaos* **20** 043130

[16] Prants W and Rech P 2013 *Physica Scripta* **88** 015001

[17] Teryokhin M T and Panfilova T L 1999 *Russian Mathematics* **43** 66

[18] Wolf A, Swift J B, Swinney H L and Vastano J A 1985 *Physica D* **16** 285

[19] Lust K 2001 *Int. J. Bifurcation Chaos* **11** 2389

[20] See, for example, the website called Glen's stuff: last accessed 10/07/2015 *Online at http://www.glensstuff.com/rosslerattractor/rossler.htm*

[21] Qi G, Wang Z and Guo Y 2012 *Int. J. Bifurcation Chaos* **22** 1250287

 ISBN: 978-0-620-70714-5