Non-universality of a constrained period doubling route to chaos for Rössler’s system

Craig Thompson, Wynand Dednam, André E. Botha

Department of Physics, University of South Africa
Edward Lorenz

\[ \begin{align*}
\dot{x} &= a(y - x) \\
\dot{y} &= x(b - z) - y \\
\dot{z} &= xz - bz
\end{align*} \]

Rayleigh-Bernard Convection

\[ a = 16, b = 45.92, c = 4 \]

E.N. Lorenz, J. Atmos. Sci. 20, 130 (1963)
Guoyuan Qi

\[ \dot{x} = a(y - x) + eyz \]
\[ \dot{y} = cx + dy - xz \]
\[ \dot{z} = -bz + xy \]

\[ a = 14, b = 43, c = -1, d = 16, e = 4 \]

G. Qi et al., Chaos Solit. Fract. 38, 705 (2008)
\[
\begin{align*}
\dot{x} &= -y - z \\
\dot{y} &= x + ay \\
\dot{z} &= b + z(x - c)
\end{align*}
\]

\[a = b = 0.2, c = 5.7\]

Period doubling
Period Doubling Bifurcations

$x = 0$

\[ y \]

\[ c \]

\[ C_1 \]

\[ C_2 \]

\[ C_3 \]
Universality of period doubling

\[ \delta = \lim_{n \to \infty} \frac{c_n - c_{n-1}}{c_{n+1} - c_n} = 4.6692 \]

\[ \alpha = \lim_{n \to \infty} \frac{d_n}{d_{n+1}} = 2.5029 \]

Sequence of periodic windows: 6,5,3, ...
$x = 0, c = 5.7$
$a = b = 0.2$
Previous work

Optimization method for finding periodic orbits:
W. Dednam and A.E. Botha, Engineering with Comp. 31, 126 (2015)

Conjecture:
For any initial condition \((x_0, y_0, z_0)\) there exists real non-zero parameters defining a Rössler system for which the solution through \((x_0, y_0, z_0)\) is periodic.

Shadowing

True trajectory

Numerical trajectory

True trajectory from $x'_0$

## Computer Assisted ‘Proof’

<table>
<thead>
<tr>
<th>$n$</th>
<th>$a$</th>
<th>$x'_0$</th>
<th>$y'_0$</th>
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Difficult case

\[
\begin{align*}
\dot{x} &= -y - z \\
\dot{y} &= x + ay \\
\dot{z} &= b + z(x - c)
\end{align*}
\]

Consider the case when \(x_0\) is large and negative, and \(y_0 = -z_0\), with \(z_0\) large and positive.

\[
z(t) = \frac{b}{c - x} + \left( z_0 - \frac{b}{c - x} \right) e^{-(c-x)t}
\]
Impossible case?
\[ x = -0.431 \]
\[ y = 0.429 \]
\[ z = 0.533 \]
Clustering
Conclusions and questions

• Hypothesis of the possible global existence of periodic orbits has prompted several new questions about a different kind of period doubling route to chaos and clustering in the parameter space.

• Pointed out a different course of possible investigation: is there still universality in period doubling routes to chaos which have always one point in common?

• What kind of bifurcations occur in this case?