The relativistic length transformation: more than a Lorentz contraction

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Abstract. A well known result of special relativity is that an object, moving with constant speed away from an inertial observer, has its proper length along the direction of observation reduced due to the Lorentz contraction. Although some might describe this effect as the relativistic length transformation, a more appropriate use of this term could apply to how the observed length of the object changes as the observer goes from the original inertial frame to a new one. Therefore the relativistic length transformation might yield an elongation or a contraction, depending on the circumstances. The general result for parallel velocities is derived. The important concept of the length transformation does not seem to be presented in this way in introductory texts. As an example of its application, the result is used to substantially simplify the derivation given in a well-known electrodynamics textbook of the relativistic transformation of the electric field, where the physical interpretation of the length transformation is obscured by the mathematics.

1. Introduction

Consider an object which has proper length L_0, as measured when it is stationary relative to the observer. If it is later moving parallel to its length at speed v, then it appears to instead have the contracted length

$$L_0\sqrt{1-\frac{v^2}{c^2}}=\frac{L_0}{\gamma_v} \qquad \ldots (1)$$

where $\gamma_v = 1/\sqrt{1-\frac{v^2}{c^2}}$ is the Lorentz factor [1]. This well-known result is of fundamental importance, but does it represent the general relativistic length transformation? The answer depends on what one means by length transformation, but this short paper aims to show that there is more to the relativistic length transformation than just the Lorentz contraction. The important concept of the length transformation does not seem to be presented in this way in introductory texts. Here the result is derived and then applied, as an example, to obtain the relativistic transformation of the electric field. The explicit use of the length transformation in this context provides a derivation which is clear on a physical basis and not obscured by copious mathematics.

2. The relativistic length transformation

Consider an observer in an inertial frame S observing a spaceship travelling with constant speed u towards a distant planet. This observer measures the length of the spaceship as L. This is not the proper length of the spaceship, since the observer is seeing it already Lorentz contracted according to

$$L = \frac{L_0}{\gamma_u} = L_0\sqrt{1-\frac{u^2}{c^2}} \qquad \ldots (2)$$

where $\gamma_u = 1/\sqrt{1-\frac{u^2}{c^2}}$ is the appropriate Lorentz factor.

For our purposes, we consider the length transformation as the relationship between the original length L as observed in this inertial frame S, and the new length $\bar{L}$ of the spaceship as observed from a new inertial reference frame $\bar{S}$ moving at speed v relative to frame S.

For simplicity we consider the new frame $\bar{S}$ to move at speed v along the direction parallel to the spaceship motion, so that all motions are along the same axis. The velocity of the spaceship relative to frame $\bar{S}$ can be obtained with the Einstein velocity addition rule [1]

$$\bar{u} = \frac{u-v}{1-uv/c^2}, \qquad \ldots (3)$$

and it therefore has length

$$\bar{L} = \frac{L_0}{\gamma_{\bar{u}}} = L_0\sqrt{1-\frac{\bar{u}^2}{c^2}}. \qquad \ldots (4)$$

The length transformation is therefore given by

$$\frac{\bar{L}}{L} = \frac{\sqrt{1-\frac{\bar{u}^2}{c^2}}}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{\sqrt{1-\frac{1}{c^2}\left(\frac{u-v}{1-uv/c^2}\right)^2}}{\sqrt{1-\frac{u^2}{c^2}}}. \qquad \ldots (5)$$

The part in the square root of the numerator, namely

$$1-\frac{1}{c^2}\left(\frac{u-v}{1-uv/c^2}\right)^2 = \frac{c^2(1-uv/c^2)^2-(u-v)^2}{c^2(1-uv/c^2)^2} = \frac{c^2\left(1-\frac{2uv}{c^2}+\frac{u^2v^2}{c^4}\right)-(u^2-2uv+v^2)}{c^2(1-uv/c^2)^2}$$

can be simplified to

$$\frac{c^2+\frac{v^2u^2}{c^2}-u^2-v^2}{c^2(1-uv/c^2)^2} = \frac{(c^2-v^2)\left(1-\frac{u^2}{c^2}\right)}{c^2(1-uv/c^2)^2} = \frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u^2}{c^2}\right)}{(1-uv/c^2)^2}.$$

Hence

$$\bar{L} = \frac{\sqrt{\frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u^2}{c^2}\right)}{(1-uv/c^2)^2}}}{\sqrt{1-\frac{u^2}{c^2}}}L = \frac{L}{\gamma_v(1-uv/c^2)}. \qquad \ldots (6)$$

The relativistic length transformation therefore consists of the Lorentz contraction factor γ_v together with the additional factor $1-uv/c^2$. This additional factor plays a key part later in the derivation given for the relativistic transformation of the electric field.

Equation 6 reduces, as it must, to the special case $\bar{L} = \frac{L_0}{\gamma_v}$ when $u = 0$. In the range $0 < v < u$ there is actually an increase in the observed length of the spaceship, until when $v = u$ the observer and spaceship are stationary relative to one another and

$$\bar{L}_{max} = \frac{L}{\gamma_u(1-u^2/c^2)} = \gamma_u L = L_0. \qquad \ldots (7)$$

This is illustrated in figure 1.

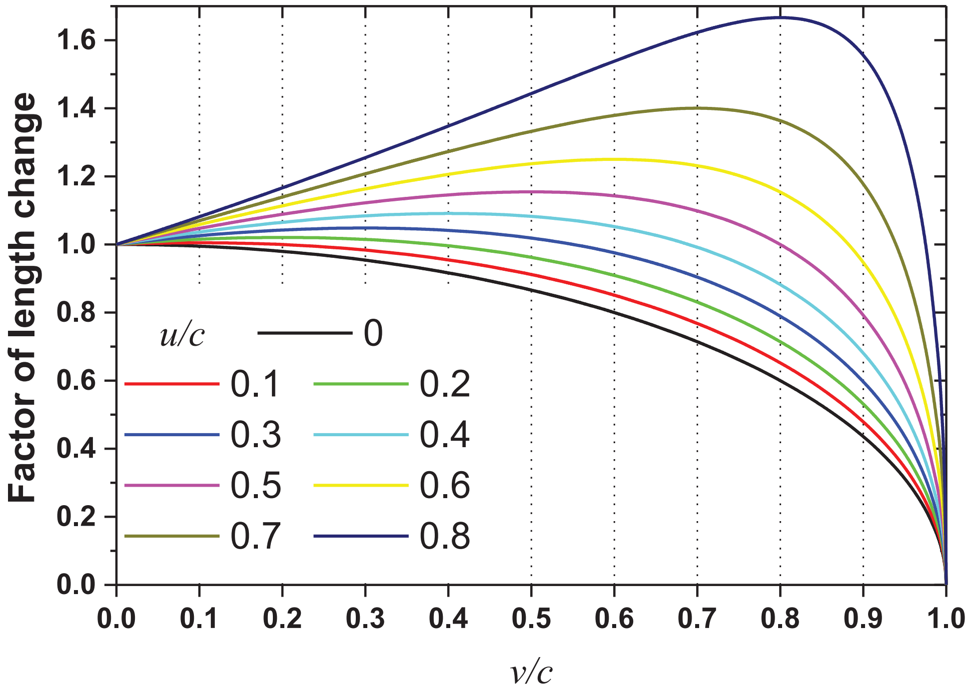

Figure 1. The factor of the length change as a function of the speed (v/c) of the new reference frame $\bar{S}$, for various speeds u/c of the object in the initial frame S.

For larger speeds a length contraction relative to the proper length L_0 occurs and there is some speed $v > u$ for which $\bar{L} = L$ again, i.e. when there is no change in the observed length despite the change in the observer's inertial reference frame. Some analysis reveals that this occurs when

$$v = \frac{2c^2}{c^2 + u^2} u \qquad \dots (8)$$

as may readily be verified by substitution into equation (6). It may be noted that an alternative manner to obtain this result is to simply set $\bar{u} = -u$ in equation 3.

3. Transformation of the electric field considered as a consequence of the length transformation

This general length transformation can, for example, be used to simply derive the transformation of the transverse component of an electric field for a change in reference frame. Consider a parallel plate capacitor with plates of equal but opposite surface densities $\pm\sigma$ lying parallel to the XZ plane and moving with speed u in the X direction. Ignoring edge-effects, this produces an electric field $E_y = \frac{\sigma}{\varepsilon_0}$ and a magnetic field $B_z = \mu_0 \sigma u$ in the region between the plates [2].

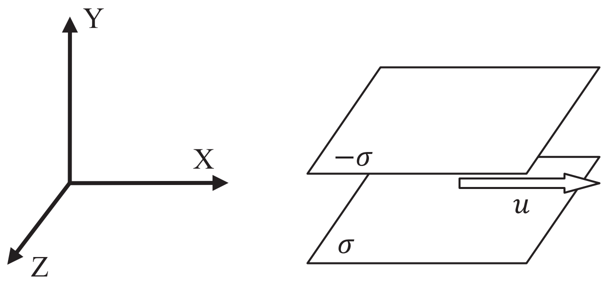

Figure 2. Moving parallel plate capacitor in original reference frame.

If the observer now moves along the X direction at speed v, there is a length transformation in the X direction i.e. contraction by the factor $\gamma_v(1 - uv/c^2)$, while the transverse Y and Z lengths are not affected. This does not affect the general geometry of the system and in the new frame

$$\bar{E}_y = \frac{\bar{\sigma}}{\varepsilon_0} = \frac{\sigma\gamma_v(1 - uv/c^2)}{\varepsilon_0} \quad \ldots (9)$$

where the surface charge density has increased by the factor of the length contraction in the X direction. Therefore $\bar{E}_y = \gamma_v \left[\frac{\sigma}{\varepsilon_0} - \frac{\sigma uv}{\varepsilon_0 c^2}\right]$. The first term in the bracket is the electric field in the original reference frame. Since $\frac{1}{\varepsilon_0 c^2} = \mu_0$ the second term is $\mu_0 \sigma uv$ and so

$$\bar{E}_y = \gamma_v\left[E_y - vB_z\right]. \quad \ldots (10)$$

The second term in the bracket, involving the magnetic field, is therefore a direct consequence of the additional factor in equation 6 besides the Lorentz contraction for the length transformation. This derivation is given, for example, by Griffiths in his textbook *Introduction to Electrodynamics* [2]. However, the length transformation is not made explicit and the derivation is therefore significantly more complicated than necessary, which detracts from the presentation and obscures the focus on the physical concepts. The length transformation is expected to be a useful concept in many problems involving special relativity.

4. Conclusion

A description of the relativistic length transformation has been provided which is more general than the well known Lorentz contraction, which may be considered as a particular case. From the derived result a change in the observer's inertial reference frame may also result in a length increase or there may be no change in the length despite the change of the observer's inertial reference frame. Use of the relativistic length transformation, where appropriate, can simplify the physics both conceptually and mathematically, which has been illustrated here for a derivation of the transformation of the transverse component of an electric field for a change in reference frame.

References

[1] Beiser A 2003 *Concepts of Modern Physics* 6th Ed (New York: McGraw-Hill)
[2] Griffiths D J 2017 *Introduction to Electrodynamics* 4th Ed (Cambridge: Cambridge University Press)