Scaling behavior of scattering observables for three-body systems near the unitary limit

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Abstract. This contribution reports recent investigations on low-energy scaling properties of three-body systems, by considering elastic $s-$wave collisions of a particle in a bound-state formed by the remaining two-body system. First, some previous results for the case of the halo nucleus ^{20}C will be revised, for the neutron$-^{19}$C scattering properties near the critical condition for the occurrence of an excited bound state in ^{20}C, within a neutron$-$neutron$-^{18}$C configuration. In our approach, we consider the Faddeev formalism with renormalized zero-range two-body interactions. Next, by considering the actual possibilities for verification of low-energy scaling properties in cold-atom laboratories, the approach is extended to atomic strongly-mass-imbalanced three-body systems, with two identical heavy particles and a light one. In this case, we consider that the heavy particle is being scattered by the light-heavy weakly-bound dimer. Our preliminary results for scattering observable are evidencing the universal scaling features.

1. Introduction

First shown by Efimov [1] for the case that we have three-identical particles in the unitary limit (more specifically, when the absolute value of the two-body scattering length $|a|$ becomes infinity), a discrete geometrical scaling emerges in the corresponding bound-state energy spectrum, which has an infinite number of three-body levels. Considered as a quantum mechanics pathology in the beginning [2], the Efimov effect was demonstrated in [3] to arise from essentially the same singularity structure of the kernel of the scattering equation, which is also responsible for the Thomas collapse [4] of the three-body ground state when the two-body range r_0 is reduced to zero. As shown in Ref. [3], the trace of the kernel will diverge as $\ln(|a|/r_0)$, allowing the Efimov and Thomas effects to be described in a unified way. The Efimov effect has been discussed occasionally along the years in the nuclear and atomic-molecular physics context [5–13] as related to some universal aspects of three-body quantum systems. The experimental observation of this effect in different ultracold laboratories [14–18], which were possible in view of the quantum-mechanical control of the two-body interactions provided by Feshbach resonance mechanisms [19], attracted lot of new theoretical and experimental investigations on low-energy quantum few-body systems. Also, the existence of the long-standing prediction of an excited Efimov state in the helium trimer [9] was reported recently in Ref. [20]. As a summary on the research activity in this topic, in atomic and nuclear physics, we should mention as relevant some previous reviews, as Refs. [21–24]. In particular, for more recent ones, we can mention Ref. [25] when considering universal aspects in light halo nuclei; and Refs. [26,27] for the case of ultracold quantum gases. For a simplified pedagogical description of the Efimov effect, we can suggest Ref. [28], where this effect is shown to be a manifestation of a well-known quantum mechanics anomaly [29].

The investigations of Efimov related effects in the scattering region is quite appealing in view of recent interest in three-body systems $\alpha - \alpha - \beta$ with non-identical masses, $m_\alpha \neq m_\beta$. In nuclear physics, the actual interest goes to weakly-bound halo-nuclei systems, with two neutrons (n) in the halo and a core (c), where one can consider the scattering of a neutron by the $n - c$ weakly-bound sub-system. With increasing possibilities to be studied experimentally, we have ultra-cold atomic systems where one heavy atom is colliding with low-energy to a weakly-bound dimer formed by the remaining two particles.

Before examining the scattering region, it is first relevant to remind that in the unitary limit, it was demonstrated that two levels of the three-body Efimov spectrum are related by an exponential scaling factor given by $\exp(2\pi/s_0)$, where s_0 is a constant that varies according to the mass-ratio m_α/m_β [23]. In the case of identical-mass system, we have the maximum energy-ratio predicted to be ~ 515, such that it will be quite difficult to be experimentally verified in laboratory, due to the large splitting of the energy levels. However, one can easily identify from the mass-dependence of the discrete scaling factor that optimal situations, better than the equal-mass cases detected for example in [16], can occur for mass-imbalanced systems with $m_\alpha >> m_\beta$, when the level splitting could be more easily experimentally identified. For instance, in case of $m_\alpha = 100 m_\beta$, the ratio between consecutive levels of the bound-state energy spectrum is given by $\exp(2\pi/s_0) \sim 4.7$. Therefore, the actual experimental investigation are more promising in this direction, by considering the atom-molecule collision with two kind of atomic species [18, 30, 31]. By following this line of investigation in cold-atom experiments, we have the Heidelberg group studying the extreme mass-imbalance atomic mixtures composed by Caesium and Lithium atomic species [27].

With the actual possibilities to manipulate Lithium(Li)-Caesium(Cs) collisions [32], as well as considering that LiCs molecules can be generated in ultracold experiments [33], we understand that more favorable conditions are accessible by now to probe the rich Efimov physics in cold-atom laboratories by investigating the low-energy collision of a Cs atom in a weakly-bound LiCs molecule. In a more general perspective, by also including other possible atomic species, we can consider the scattering of an atomic specie α in a weakly-bound $\alpha - \beta$ system, for $m_\alpha >> m_\beta$. For that, in our following theoretical approach we assume that the two-body interactions are such that there is no interaction between the two heavy particles. Further, by controlling the strength of the $\alpha - \beta$ binding in laboratory through Feshbach resonance mechanism, we understand that appropriate conditions can be created for atom-molecule collisions at very low energies.

Within our approach to identify the scaling behavior of scattering observables for three-body systems near the unitary limit, let us recall some recent discussion related to the $k \cot \delta_0^R$ pole behavior in the weakly bound three-body nuclear physics case of carbon-20 (^{20}C), within the three-body neutron-neutron$-^{18}$C model, where the subsystem neutron$-^{18}$C is bound. In this case k is the on-energy shell momentum (incoming and final), with δ_0^R being the real part of the $s-$wave phase shift. This case, with two light particles and a heavy one, was shown to resemble the well-known equal-mass neutron-deuteron case, which was studied about forty years ago [34–36]. In contrast with these studies, when considering the present experimental possibilities in cold-atom laboratories, we can verify that the limit of two-heavy and one light particle is quite more interesting to be explored near the unitary limit. As in this limit, for the case $m_\alpha >> m_\beta$, we can identify several bound-state poles for the three-body spectrum with levels close together, in correspondence, a sequence of poles is expected to appear in the scattering observable $k \cot \delta_0^R$ for low-energy collisions of the heavy particle in the $\alpha - \beta$ bound sub-system, which can be identified by the corresponding minima in the cross sections.

In the next we have two sections reporting results for scattering observables in mass-asymmetric cases. Section 2 is concentrated in presenting some results and analysis related do the halo-nucleus ^{20}C, with section 3 having its focus in a more general mass-imbalanced atomic system with two heavy and one light particle. Section 4 gives some concluding remarks.

2. Universal properties in Neutron$-^{19}$C scattering

The low-energy properties of the elastic $s-$wave scattering for the $n-^{19}$C near the critical condition for the occurrence of an excited Efimov state in $n-n-^{18}$C was studied in Refs. [37,38], by considering renormalized zero-range as well as finite-range interactions. By fixing the two-neutrons separation energy in ^{20}C with available experimental data, it is studied the scaling of the real (δ_0^R) and imaginary parts of the $s-$wave phase-shift with the variation of the $n-^{18}$C binding energy. Universal characteristics are identified for the pole-position of $k\cot(\delta_0^R)$ and effective-range parameters. It was verified that the excited state of ^{20}C goes to a virtual state when increasing the $n-^{18}$C binding energy, resembling the neutron-deuteron behavior in the triton.

The basic formalism for a three-body halo nucleus formed by a core and two neutrons interacting via two-body separable potential is presented in Refs. [37,38]. The $n-n$ interaction was considered fixed, such that it was obtained the usual virtual-state energy, $E_{nn} = -143$ keV. For the $n-c$ subsystem ($E_{nc} = E_{^{19}\mathrm{C}}$), the bound-state energies are varied within a range given by experimental available data, ranging from -160 ± 110 keV [39] to -530 ± 130 keV [40]. The $n-n-c$ three-body ground-state binding energy is fixed at $E_{^{20}\mathrm{C}} = -\ 3.5$ MeV. The units for this case are such that $\hbar = 1$, with momentum variables in fm^{-1} and the unit conversion given by $\hbar^2/m_n = 41.47$ MeV fm^2. The core-mass number is defined as $A = m_c/m_n$, with $\mu_{nc} = Am_n/(A+1)$ being the reduced mass for the $n-c$ system, such that we have $A = 18$ and $\mu_{nc} = (18/19)m_n$ in the specific system we are considering.

The zero-range two-body interaction considered for the $s-$wave elastic $n-(nc)$ scattering formalism in Ref. [37] was extended to include finite-range two-body interactions, with separable form, given by

$$V_{ij}(p,p') = \lambda_{ij}\left(\frac{1}{p^2+\beta_{ij}^2}\right)\left(\frac{1}{p'^2+\beta_{ij}^2}\right), \tag{1}$$

where $ij = nn$ or nc, respectively, for the $n-n$ or $n-c$ two-body subsystems. λ_{ij} and β_{ij} refer to the strength and range r_{ij} of the respective two-body interaction. For negative bound or virtual two-body energies E_{ij}, the corresponding relations for the strengths and ranges are

$$\lambda_{ij} = \frac{-2\pi\mu_{ij}}{\beta_{ij}(\beta_{ij}\pm\kappa_{ij})}, \qquad r_{ij} = \frac{1}{\beta_{ij}} + \frac{2\beta_{ij}}{(\beta_{ij}\pm\kappa_{ij})^2}, \tag{2}$$

where $\kappa_{ij} = \sqrt{-2\mu_{ij}E_{ij}}$, with (+) for bound and (−) for virtual states.

By following the formalism given in Ref. [38] for the on-shell scattering amplitude $h_n(k;E)$, where E is the three-body energy $E \equiv E(k_i)$, we obtain:

$$h_n(k;E) = \frac{e^{\mathrm{i}\delta_0}\sin\delta_0}{k} = \frac{1}{k\cot\delta_0 - \mathrm{i}k}, \tag{3}$$

with the on-shell relative momentum given by $k \equiv k_i \equiv |\vec{k}_i| = |\vec{k}_f| = \sqrt{2\mu_{n(nc)}\,(E-E_{nc})}$ and the reduced mass is $\mu_{n(nc)} = m_n(A+1)/(A+2)$.

The validity of the picture found with the renormalized zero-range model in Ref. [37] is verified with finite-range interactions in [38] compatible with the nuclear force. The analysis follow a suggestion given in Ref. [34] for a parameterization of the effective-range expansion of $k\cot\delta_0^R$, where δ_0^R is the real part of the corresponding $s-$wave phase shifts. With E_0 being the pole position for the energy, we have

$$k\cot\delta_0^R = \frac{-a^{-1} + b\,E + c\,E^2}{1 - E/E_0}, \tag{4}$$

where a [$n-(nc)$ scattering length], b and c are the adjustable parameters. When considering short-ranged interactions, such that we have fixed the two-neutron separation energy in ^{20}C (3.5 MeV [39]) and the $n-n$ virtual state energy ($E_{nn} = -143$ keV), as the ^{19}C binding energy is varied from 0.2 to 0.8 MeV, the corresponding parameters are given in the following Table 1, where r_{nn} and r_{nc} are the corresponding ranges.

Table 1. One-neutron separation energy in ^{19}C, obtained by the given low-range parameters of the separable potential. The values of β_{nc} were obtained by fitting the two-neutron separation energy in ^{20}C (3.5 MeV [39]), with the $n-n$ interactions fixed by the virtual state energy, $E_{nn} = -143$ keV. In this case, we fix β_{nn} =24.5 fm^{-1} (r_{nn} =0.1228).

$\lvert E_{^{19}\mathrm{C}}\rvert$(keV)	200	400	600	800	830
β_{nc}(fm^{-1})	18.970	17.036	15.592	14.395	14.234
r_{nc}(fm)	0.157	0.174	0.190	0.205	0.207

The parameters for the low-range potential shown in Table 1 provide results consistent with the ones obtained in Ref. [37] when using renormalized zero-range interactions. In Fig. 1 we present the results obtained by considering short-range interactions, reproduced from Ref. [38]. From this Fig. 1, by a fitting procedure, one can obtain the values for the parameters of the effective-range expansion (4). The corresponding parameters are shown in Table 2.

Table 2. Effective-range parameters, obtained by fitting Eq. (4) to Fig. 1, when considering different values of $\lvert E_{^{19}\mathrm{C}}\rvert$ (first column) with short-range potentials.

$\lvert E_{^{19}\mathrm{C}}\rvert$ (keV)	$-1/a$ (fm^{-1})	$10^4\,b$ (fm.keV)$^{-1}$	$10^8\,c$ (fm.keV^2)$^{-1}$	E_0 (keV)	d (fm^{-1})	r_{nc} (fm)
200	0.006	5.579	5.717	1304	0.831	0.157
400	0.066	6.742	9.144	749.0	0.622	0.174
600	0.234	9.840	13.16	402.9	0.652	0.190
800	1.798	44.67	35.19	78.86	2.153	0.205
830	5.149	119.8	85.78	28.98	5.497	0.207

Finally, we observe that, in analogy with the known behavior of the elastic neutron-deuteron doublet $s-$wave scattering amplitude, the results presented in [37,38] confirm that the real part of the elastic $s-$wave phase shift (δ_0^R) reveals a zero when the $n-n-c$ system is close to an excited Efimov state (bound or virtual). For more details on this investigation, see Ref. [38], where the zero-range results are being compared with finite-range ones, by using separable one-term two-body interactions with different parametrizations. In this reference, results for the $s-$wave absorption parameter are also presented. Our expectation is that a study with more realistic potential models (following the approach of Ref. [41]), will confirm the main conclusions on the universal properties verified for the $n-^{19}$C scattering in the models used in Refs. [37,38].

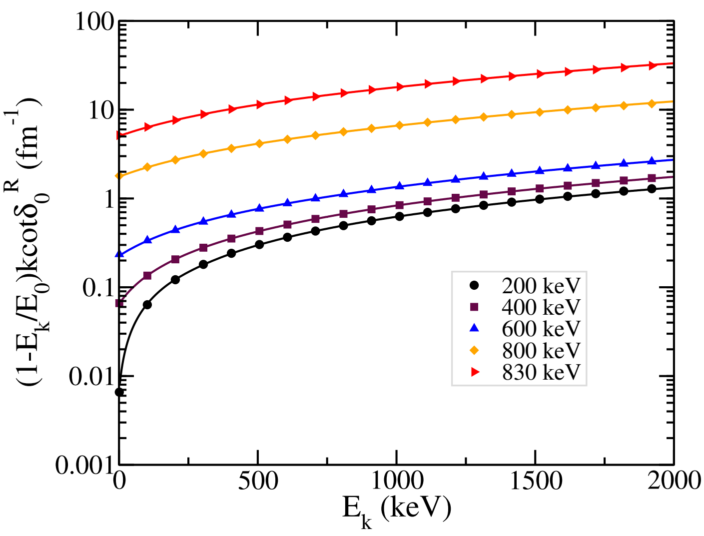

Figure 1. The function $(1 - E_K/E_0)k \cot \delta_0^R$, where $E \equiv E_K$ and E_0 is the pole position, is given in terms of $E_K \equiv k^2/(2\mu_{n,nc})$, by considering one-term separable low-range potentials (large β values), shown in Table 1. For each case, we plot the results obtained for different values of the two-body binding energy $|E_{nc}|$, as indicated inside the frame (adapted from Ref. [38]).

3. Scaling behavior in atom-dimer $\alpha - (\alpha - \beta)$ scattering with $m_\alpha \gg m_\beta$

As motivated in our introduction, large mass imbalanced systems can better be investigated in cold-atom experiments in view of possibilities to alter the two-body scattering length by tuning a Feshbach resonance with magnetic fields. Therefore, it is interesting to extend the approach considered in section 2 to the case of the collision of a heavy atom α on a dimer formed by α and a lighter atom β, in the case that $m_\alpha \gg m_\beta$. For this kind of system, one can also consider a Born-Oppenheimer (BO) approach in addition to the basic Faddeev three-body formalism that was used for the $n - (n - ^{18}$ C) system, and check for the properties of the s-wave elastic cross section, while keeping the essence of the Efimov physics.

For that aim, we have examined the low-energy collision of a heavy particle α with a weakly-bound two-body $\alpha - \beta$ system. The heavy-heavy interaction is assumed zero as this should not affect the driving Efimov physics, which is originated by the heavy-light resonant interaction as already pointed out long ago by Fonseca and Shanley [7]. In this case the Efimov physics of the system is driven by the mass ratio and the ratio between the weakly-bound energy of the two-body system $B_{\alpha\beta}$ and a three-body energy scale, which can be chosen as the three-body bound state energy [25]. The two-body bound-state energy can also be represented by the corresponding very-large positive two-body scattering length, which is given by $a \equiv a_{\alpha\beta} = \sqrt{(2\mu_{\alpha\beta}/\hbar^2)/B_{\alpha\beta}}$, where $\mu_{\alpha\beta}$ is the reduced mass between α and β particles.

In this case, for a very large mass ratios $m_\alpha \gg m_\beta$, namely for our purpose we choose $m_\alpha/m_\beta = 20$ we can show that the minimum of the s-wave elastic cross-section appears and more than one. These minima are associated with the poles of $k \cot \delta_0^R$, which are associated with the long range Efimov potential, strengthen by the mass ratio effect that implies a smaller geometrical ratio between the Efimov states as larger is the mass asymmetry in the heavy-heavy-light system.

Our results [42], which are verified for several small values of $B_{\alpha\beta}$, by using Faddeev approach as well as a BO approximation, are being exemplified in two panels of Fig. 2 for the cases that the mass-ratios is given by $A = m_\beta/m_\alpha = 0.08$ and 0.05, when considering $B_{\alpha\beta} = 0.05\mathcal{B}_3$, where $\mathcal{B}_3$ is an energy scale that we assume equal to the three-body ground-state energy. The novel feature emerging in these two panels is verified by the possibility that more zeros can be identified in the cross-section (or more poles in the $k\cot\delta_0^R$), as the mass asymmetry increases. This observation is closely related to the dominance of the Efimov long-range potential, which becomes more visible as the mass asymmetry increases. In a semi-classical description such zeros should come as a consequence of the interference between different classical trajectories that passes around the two sides of the bound target. Therefore, it is conceivable that more zeros of the $s-$wave cross-section should appear as the intensity of the Efimov long range potential raises.

As a final remark on this case, we should point that the ratio between the positions of the scattering momentum $k^{(n)}$ of the two-zeros shown in the upper plot of Fig. 2, for $A = 0.05$, is given by $\sqrt{E^{(n)}/E^{(n+1)}} = 6.5$, when the two-body energy is $B_{\alpha\beta} = 0.05\mathcal{B}_3$. This result, when compared with the corresponding Efimov ratio (note that $\sqrt{B_3^{(n)}/B_3^{(n+1)}} \approx 5$ for $B_{\alpha\beta} = 0$, in this mass asymmetric case), strongly suggests the plausibity of the above semi-classical description.

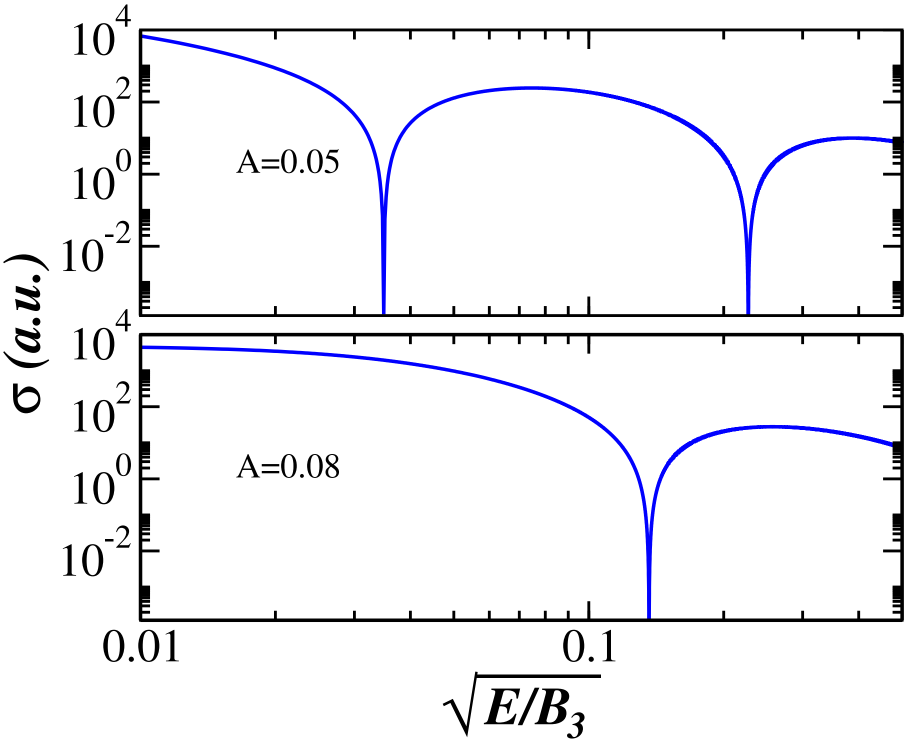

Figure 2. Results obtained for the total $s-$wave cross-section σ (in arbitrary units) as a function of $\sqrt{E/\mathcal{B}_3}$, by considering the mass-ratios $A \equiv m_\beta/m_\alpha$ =0.05 and 0.08, with $B_{\alpha\beta}/\mathcal{B}_3 = 0.05$. As shown, by decreasing the mass ratio A (from lower to upper panels), we can already identify the emergence of a second minimum in σ. For A =0.05, the ratio between the scattering momenta where the two minima are ocurring (with $B_{\alpha\beta}/\mathcal{B}_3 = 0.05$), which is ~ 6.5, can already be compared with the ratio between the Efimov levels in the three-body bound-state spectrum (where $\sqrt{B_3^{(n)}/B_3^{(n+1)}} \approx 5$ for $B_{\alpha\beta} = 0$).

4. Concluding remarks

In the introduction we give the motivations to study mass-asymmetric systems in nuclear and atomic cases, and following it in section 2, we shortly review results already obtained on the low-energy properties of the elastic $s-$wave neutron$-^{19}$C elastic scattering near the critical condition for the occurrence of an excited Efimov state. The analysis of that results was done in close analogy with the case of the $s-$wave neutron-deuteron doublet phase-shift, where it is well known that at low-energies the properties of the neutron-deuteron doublet state is dominated by the Efimov physics, and the parameters of the effective range expansion of the neutron-deuteron doublet $s-$wave phase-shift present universal scaling laws with the triton binding energy for fixed nucleon-nucleon scattering lengths and effective ranges.

Next, by following the approach considered for the neutron$-^{19}$C scattering, it was communicated some examples on large mass-asymmetric cases of interest in recent cold-atom investigations. The low-energy s-wave elastic cross-section results were obtained for the collision between a heavy atom α and a weakly-bound heavy-light $(\alpha - \beta)$ sub-system. These studies are being motivated by the actual experimental possibilities, along the lines as reviewed in Ref. [27]. We should also point out the relevance of an extension of the present investigation to heteronuclear ultracold quantum gases.

As being verified, the scattering observable $k \cot \delta_0^R$ presents poles, corresponding to the zeros of the $s-$wave cross section, which can be directly connected with the Efimov physics, as observed when approaching the unitary limit. Here we have presented a sample of results for the mass-imbalanced system, considering $m_\beta/m_\alpha = 0.08$ and 0.05 in the case that $B_{\alpha\beta} = 0.05\mathcal{B}_3$. As shown, a second pole in $k \cot \delta_0^R$ can be found as we increase the imbalance between the masses of the components of the three-body system. A more detailed investigation is in progress.

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