Construction of lambda-nucleon s-wave potential through quantum inverse scattering at fixed angular momentum

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Abstract. Quantum systems with a strangeness degree of freedom are very important as they provide an extra dimension, and hence a deeper insight into nuclear matter. Usually phenomenological potentials obtained through meson exchange theories are used in investigating these hypernuclear systems. In this paper potentials for lambda-nucleon interactions in the spin singlet and spin triplet states, constructed through fixed-angular momentum inversion based on Marchenko theory, are presented. Owing to experimental difficulties in obtaining a sufficient number of lambda-nucleon scattering events, theoretical phase shifts are used as input for the inversion. The constructed potential is energy-independent, making it more suitable for quantum-mechanical few-body calculations.

1. Introduction

The choice of a suitable baryon-baryon potential for use in simulations has been one of the preoccupations right at the heart of nuclear physics for much of its history. In particular, the last two or three decades has seen an increased effort towards understanding baryon-baryon interactions for systems with a strangeness degree of freedom. Much of the effort has been directed towards single ($S = -1$) and double ($S = -2$) strangeness systems. These potentials are important for many reasons. For example, they are used as input in simulations of hypernuclei. The glue-like role of hyperons within hypernuclei and the nonmesonic decay of hyperons in large-baryon number systems through the weak interaction are phenomena requiring further elucidation that may only come from more reliable hyperon-nucleon potentials. These interactions also provide a deeper understanding of multistrangeness systems such the core of a neutron star, whose equation of state requires hyperon-nucleon and hyperon-hyperon potentials.

Meson-exchange theories have been the principal framework for developing most of the hyperon-nucleon and hyperon-hyperon potentials currently in use for few-body calculations. The most widely used meson-exchange potentials are the soft-core versions of the Nijmegen potentials (Rijken *et al.* [1]), which were first formulated in de Swart *et al.* [2]. A historical review of the development of the Nijmegen potentials is found in de Swart *et al.* [3] and Hiyama *et al.* [4]. While most of these potentials do successfully address certain important issues such as the overbinding problem, they still suffer some inadequacies. The sensitivity of few-body calculations to the choice of potentials is symptomatic of these inadequacies, within the error limits of the few-body method used. Could these inadequacies stem from a much wider problem of nuclear physics using potentials constructed between structureless hadrons, even at energy scales for which the quark degrees of freedom in Quantum Chromodynamics (QCD) represent the "correct" physics? Already, there has been considerable effort in building new potentials based on QCD. The Kioto-Niigata potential for all interactions between spin-1/2 octet baryons (n, p, Σ^+, Σ^0, Σ^-, Ξ^-, Ξ^0 and Λ) is one such development (Nakamoto *et al.* [5], Fujiwara *et al.* [6] - [7]). However, these quark-model potentials have not enjoyed widespread applications in few-body calculations as the meson-exchange potentials. A more systematic inclusion of QCD theories in nuclear physics *may* hold more for the future of nuclear physics, as QCD is the underlying theory of the strong interaction. The ambiguities of the currently used hypernuclear potentials make it necessary to investigate potentials obtained using a different theory.

In this paper we propose new potentials for the ΛN interaction in the spin singlet and triplet states, constructed not using QCD theories, but through quantum inverse scattering theory. The fixed-angular momentum approach has been used in solving this inverse scattering problem. Owing to its simplicity when compared to Gel'fand-Levitan inversion, the Marchenko inversion scheme is used. The rest of the paper is organised as follows: Sections 2 is a brief recapitulation of quantum inverse scattering at fixed angular momentum, Section 3 is devoted to ΛN scattering experiments and theoretical scattering data, while Sections 4 and 5 contain the results and conclusion, respectively.

2. The scattering matrix to potential inverse problem

2.1. Marchenko theory

Inversion may either be performed from experimental observables to the potential, or from the scattering matrix to the potential. The inversion from scattering matrix to potential may either be done at a fixed angular momentum (ℓ) or at a fixed energy (k), or using a hybrid formalism for a given energy range and set of angular momenta. The usual normalisation $\hbar^2\psi_l/2\mu = \psi_l$ is used. A detailed review on inverse scattering theory is found in the monograph by Chadan and Sabatier [8], while an up-to-date review on applications in nuclear physics may be found in Kukulin and Mackintosh [9]. Of interest in this paper is Marchenko inversion, one of the fixed-angular momentum formalisms. For a short-range potential, $V_\ell(r)$, with spherical symmetry the potential for the scattering matrix at each partial wave ℓ is constructed by solving the following Fredholm integral equation for $K_\ell(r,r')$:

$$K_\ell(r,r') + A_\ell(r,r') + \int_r^\infty K_\ell(r,s)A_\ell(s,r')ds = 0 \tag{1}$$

Here $A_\ell(r,r')$ is a symmetric input kernel that must be known. This input kernel is constructed as in Fiedeldey *et al.* [10], taking into account any bound states:

$$A_\ell(r,r') = \frac{1}{2\pi}\int_{-\infty}^{\infty} h_\ell^+(k,r)\left\{1 - S_\ell(k)\right\} h_\ell^+(k,r')dk + \sum_{i=1}^{n_b} M_i h_\ell^+(k_i,r)h_\ell^+(k_i,r') \tag{2}$$

where n_b is the number of physical bound states, if there are any, and $h_\ell^+(k,r)$ are Riccati-Hankel functions. The M_i are normalisation constants for the bound-state wavefunctions. The unphysical bound states, the Pauli Forbidden States, which may arise are usually removed through supersymmetry.

In inversion theory $K_\ell(r,r')$ arises as the kernel of a transformation from the solution of Schrödinger equation for a free particle to any solution of the radial Schrödinger equation, for example the Jost solutions (Levin [11], Marchenko [12]). Furthermore, this output kernel, which has the property of strict upper triangularity ($K_l(r,r') = 0, r > r'$), satisfies a Goursat problem which is right at the heart of quantum inverse scattering (Gel'fand and Levitan [13], Newton and Jost [14], Agranovich and Marchenko [15], Levitan [16], Deift and Trubowitz [17], Newton [18]). The auxiliary condition of this Goursat problem on the characteristic curve $r = r'$ ensures that the diagonal entries in the kernel are read off as the scattering potential i.e.

$$-2\frac{d}{dr}K_\ell(r,r) = V_\ell(r) \tag{3}$$

Therefore, solving the inverse problem of scattering matrix to potential can be understood in proper mathematical terms as reconstructing the auxiliary conditions of this Goursat problem.

Marchenko inversion theory has been used to construct particle-particle potentials, in particular nucleon-nucleon potentials (Benn and Scharf [19] - [20], Coz *et al.* [21], Kirst *et al.* [22], Sparenberg and Baye [23]). Kukulin and Mackintosh [9] contains an extensive list of references for particle-cluster and cluster-cluster potentials that have been constructed through inversion theory, for example the neutron-alpha potential.

Here, we apply Marchenko inversion theory to construct ΛN potentials. A motivating factor for such an attempt stems from the fact that the widely used models for the ΛN potential disagree with experimental results in terms of single lambda binding energies (B_Λ) of ground state and excited states, even for low-baryon number hypernuclei.

2.2. Regularisation of linear system

Theory requires that $S_\ell(k)$, which is a complex-valued function, should be known for all energies or wavenumbers i.e. $k \in [0, \infty)$. However, the S-matrix is obtained from the phase shift, which is itself derived from experimental observables: $S_\ell(k) = \exp(2i\delta_\ell(k))$, where $\delta_\ell(k)$ is the phase shift. The energy range of these experimental observables are limited by laboratory technicalities, thus limiting the energy range for which $S_\ell(k)$ is known. With such a scattering matrix, the inverse problem of computing the potential from the scattering matrix results in a linear system in which there are more degrees of freedom than there are constraints to restrict these degrees of freedom. Such an underdetermined linear system has no solution, or if it is homogeneous it has infinitely many solutions.

In order to solve this inverse problem, some form of regularisation is needed, so as to provide additional information. The most common approach, which is also the one that has been used in this paper, is to interpolate $S_\ell(k)$ using a rational function, and then extrapolating it to sufficiently higher energies. Making use of the Bargmann rational-function representations of the Jost functions (Chadan and Sabatier [8]), the following interpolation, which explicitly shows the singularities of the Jost functions, is used:

$$S_\ell(k) = \prod_{n=1}^{N} \left(\frac{k+\alpha_n^\ell}{k-\alpha_n^\ell}\right)\left(\frac{k-\beta_n^\ell}{k+\beta_n^\ell}\right) \tag{4}$$

Here α_n^ℓ are N simples situated both in the upper-half and lower-half k plane, while β_n^ℓ are N simple poles situated only in the lower-half k plane (Massen *et al.* [24], Rakityansky *et al.* [25]). For uniformity, one may use the same symbols to represent all the roots and singularities of the S-matrix. In this case the parametrization takes the following form:

$$S_\ell(k) = \prod_{n=1}^{M} \frac{k+\alpha_n^\ell}{k-\alpha_n^\ell}, \quad \text{where} \quad M = 2N \tag{5}$$

This parametrisation ensures conservation of probability current (unitarity of $S_\ell(k)$). It also results in an exact solution to the Fredholm integral equation, as its kernel becomes degenerate; no quadrature is therefore needed (Sofianos *et al.* [26]). Kirst *et al.* [22] carries an outline of the procedure for transforming the integral equation into a linear system.

3. ΛN elastic scattering

There is very little data on ΛN scattering from experiments. This is because of the difficulty in using free hyperons as projectiles or targets in these experiments, owing to their very short lifetimes (about 10^{-10} s). When compared to free protons that do not decay, the extremely short lifetime of free hyperons poses enormous difficulties. Cross sections for the Λp elastic scattering reaction

$$\Lambda + p \to \Lambda + p \tag{6}$$

have been reported for laboratory-frame Λ momenta up to a few hundreds of GeV/c. Most of the Λp scattering data obtained have large error bars, and a low number of scattering events, insufficient to constraint the ΛN interaction. The use of such high-momentum Λ beams has significantly increased the Λ decay lengths from a few millimetres to several centimetres, thereby increasing the chances of scattering events in recent experiments. However, the higher the incident momenta, the greater the number of partial waves that must be accounted for; usually, only the s-waves suffice at low incident momenta. In an experimental set up Λn scattering is not as manageable as Λp scattering, due to the inexistence of free stable neutron targets. The lifetime of a free neutron is only approximately 881.5 s.

The link between quantum scattering theory and two of the observables that are usually measured in scattering experiments, differential cross section ($d\sigma/d\Omega$) and integrated cross section (σ), is established by the scattering amplitude, $f_k(\theta)$. The scattering amplitude accounts for the distortion suffered by the incoming wave after scattering. In terms of Partial Wave Analysis, the scattering amplitude may be written in the Faxen-Holtzmark formalism (Joachain [27]) as

$$f_k(\theta) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2l+1)(S_\ell(k)-1)P_\ell(\cos\theta) \tag{7}$$

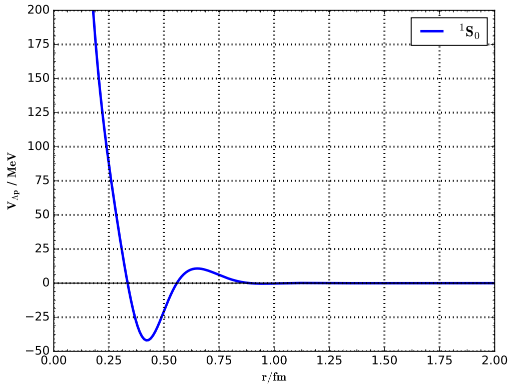
where P_ℓ is a Legendre polynomial at scattering angle θ. Solving the Radial Schrödinger Equation, one gets the phase shift, which is directly related to the scattering amplitude and cross sections. Using phase shift analysis, the contribution of each partial wave may be independently extracted. A poor phase shift analysis may therefore compromise the inversion procedure. This is an issue which is not of interest in this paper.

For our construction we have as input theoretical $^1S_0$ and $^3S_1$ phase shifts computed by Rentmeester and Klomp [28] using the NSC97f potential. Even after NN elastic scattering data became widely available, theoretical NN data continued playing an important role in our understanding of the NN force. It is hoped that this strategy will throw more light on the nature of baryon-baryon interactions in the single strangeness sector.

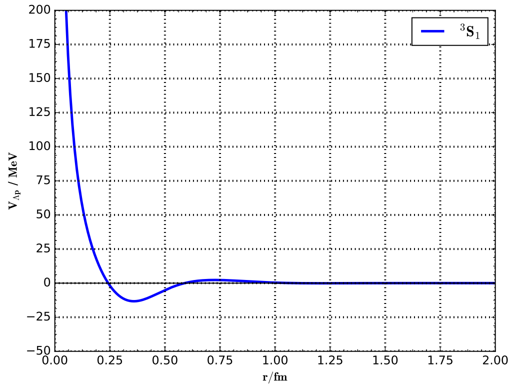
4. Results

The results are shown for the s-wave Λp (Figures 1(a) and 1(b)) and Λn (Figures 2(a) and 2(b)) potentials, both in the $S = 0$ and $S = 1$ states. The general features of a baryon-baryon interaction are present in these potentials, except for the $\Lambda n(^3S_1)$ potential where the short range repulsion is absent. The well-known result, that the $\Lambda N(^3S_1)$ potential is weaker than the $\Lambda N(^1S_0)$ potential, is also verified. However, an important feature of these potentials is that the strongest attraction occurs at a smaller radial distance than with most other lambda-nucleon potentials.

Poor choices of $S_\ell(k)$ parametrisation have been known to result in unphysical oscillations in inversion potentials, even in cases where phase shifts from experiments are used (Howell *et al.* [29]). These oscillations are absent in the results obtained here. However, the presence

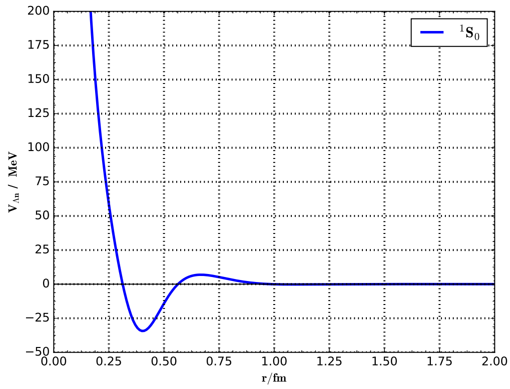


(a) Λp potential in $^1\mathrm{S}_0$ state.

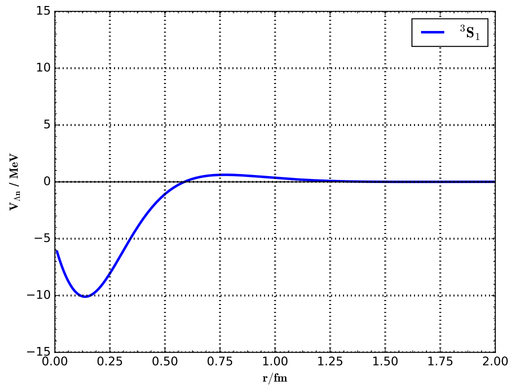


(b) Λp potential in $^3\mathrm{S}_1$ states

Figure 1



(a) Λn potential in $^1\mathrm{S}_0$ state.



(b) Λn potential in $^3\mathrm{S}_1$ states

Figure 2

of a small repulsion barrier, which is quite negligible in the spin triplet states, *may* still be a pathology arising from the parametrisation. Since Λ is a neutral elementary particle, this barrier could not possibly be a Coulomb repulsion. Further investigation is needed to clarify the effects of various $S_\ell(k)$ interpolations. This computational artefact is *not* expected to detract from the overall usefulness of our potential. This may be better appreciated in light of the inadequacies of the current potential models used in quantum-mechanical few-body calculations.

5. Conclusion

New spin singlet and spin triplet state potentials for the Λp and Λn interactions have been constructed for $\ell = 0$ through Marchenko theory, a quantum inverse scattering formalism. These potentials are energy-independent, a feature that makes them ideal for few-body calculations.

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