

Partial widths of a multi-channel resonance

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Abstract. A new method for finding the partial decay-widths for multi-channel resonances is suggested. It is based on using the Jost matrices. These matrices can be either calculated (when the multi-channel potential is known) or found by fitting experimental data (when they are available). In both cases the procedure for obtaining the partial widths is the same and is simple. It does not require any kind of integration or differentiation. The partial widths sum up to the total width and the corresponding branching ratios can be found as some algebraic combinations of the elements of the Jost matrices.

1. Introduction

In the quantum world, practically all scattering processes are multi-channel. This means that a collision of the quantum systems A and B may result in many different final products (channels)¹. At certain collision energies, such a process goes via the formation of an intermediate quasi-bound state R which has a finite lifetime. This unstable state is a quantum resonance. It decays into all open channels with different probabilities,

$$A + B \longrightarrow R \longrightarrow \begin{cases} A + B , \\ C + D , \\ E + F , \\ \text{etc .} \end{cases} \quad (1)$$

After its formation, the resonance may randomly decay in any direction, irrespective of where the original systems A and B came from, i.e in a sense it “forgets” how it was created. Mathematically, this means that the asymptotic behaviour of the corresponding wave function involves only the outgoing waves.

The half-life $T_{1/2} = (\hbar/\Gamma) \ln 2$ of the resonance R is determined by its total width Γ . If at the initial instant of time, $t = 0$, the number of generated resonances is $N(0)$, then it exponentially diminishes as

$$N(t) = N(0) \exp\left(-\frac{\Gamma}{\hbar}t\right) . \quad (2)$$

For each channel n , there is its own width Γ_n , which is called partial width. It is easy to see that the partial widths are fractions of the total width and must add up to Γ . Indeed, for several concurrent decay processes (1), the function $N(t)$ is a solution of the following differential equation:

$$\frac{dN}{dt} = -\frac{\Gamma_1}{\hbar}N - \frac{\Gamma_2}{\hbar}N - \dots = -\frac{1}{\hbar}(\Gamma_1 + \Gamma_2 + \dots)N , \quad (3)$$

¹ We restrict our consideration to two-body channels.

from where it follows that

$$\Gamma = \Gamma_1 + \Gamma_2 + \dots . \quad (4)$$

There are many different ways of locating resonances (see, for example, Ref. [1]). The most rigorous and accurate of them are those which are based on the fact that each resonance corresponds to a pole of the S -matrix at a complex energy,

$$\mathcal{E} = \mathcal{E}^{(r)} - \frac{i}{2}\Gamma . \quad (5)$$

All the existing methods are able to find the resonance energy, $\mathcal{E}^{(r)}$, and the total width to a good accuracy. The disagreements among the methods show up when it comes to the partial widths.

There are two big groups of different approaches to finding the partial widths. The methods of the first group (see, for, example, Ref. [2]) are based on the ideas similar to the so-called ‘‘Fermi golden rule’’ which can give the amplitude of the transition from the resonant state $|R\rangle$ to a continuum state $|\psi_n\rangle$ in the channel n . Within such an approach, the corresponding partial width can be found as

$$\Gamma_n = 2\pi |\langle \psi_n | V | R \rangle|^2 , \quad (6)$$

where V is the interaction potential. The partial width defined as in Eq. (6), can also be obtained from the residue of the S -matrix,

$$S(E) = 1 - 2\pi iT(E) = 1 - 2\pi i [V + V(E - H)^{-1}V] ,$$

at the pole when $E \rightarrow \mathcal{E}$. Writing the propagator in the form of the Berggren spectral expansion [3],

$$(E - H)^{-1} = \sum_b \frac{|b\rangle\langle b|}{E - \mathcal{E}_b} + \sum_q \frac{|q\rangle\langle q|}{E - \mathcal{E}_q} + \mathcal{G}(E) , \quad (7)$$

which involves singular terms corresponding to bound states $|b\rangle$ and quasi-bound (resonant) states $|q\rangle$ as well as a background term $\mathcal{G}(E)$, we see that near a resonant pole, the dominant contribution to the S -matrix comes from a single term,

$$\langle \psi_n | S(E) | \psi_{n'} \rangle \xrightarrow{E \rightarrow \mathcal{E}} -2\pi i \frac{\langle \psi_n | V | R \rangle \langle R | V | \psi_{n'} \rangle}{E - \mathcal{E}^{(r)} + \frac{i}{2}\Gamma} = \frac{-i\sqrt{\Gamma_n \Gamma_{n'}}}{E - \mathcal{E}^{(r)} + \frac{i}{2}\Gamma} , \quad (8)$$

and the absolute value of the residue coincides with $\sqrt{\Gamma_n \Gamma_{n'}}$. It was noted in many publications (see, for example, Refs. [4, 5]) that within this ‘‘golden rule’’ approach the partial widths obey Eq. (4) only for narrow resonances, i.e. for short-lived states they do not sum up to the total width. In Sec. 2 and 4, it is given a possible explanation of such a discrepancy.

The second group unites the methods, where the partial widths are extracted from the asymptotic behaviour of the resonant state (at large distance r between the decay fragments),

$$|R\rangle \xrightarrow{r \rightarrow \infty} \begin{pmatrix} A_1 |\psi_1\rangle \\ A_2 |\psi_2\rangle \\ \vdots \end{pmatrix} . \quad (9)$$

In this asymptotics the amplitude A_n determines the probability of finding the system in the n -th channel and thus is related to Γ_n (see, for example, Ref. [6]).

Our approach belongs to the second group. The novelty of our method mainly consists in the way of finding the amplitudes A_n . We evaluate them using the multi-channel Jost matrices. These matrices can either be calculated from a given potential or fitted to any available experimental data. After obtaining the Jost matrices, the partial widths are calculated as simple algebraic combinations of their matrix-elements. This procedure does not involve any integration or differentiation and does not require ‘‘proper’’ normalization of the resonance wave function.

2. Partial widths in an experiment

Experimental equipment is always a macroscopic object. Typical distances traveled by the decay products before they are detected, are many orders of magnitude greater than the size of the quasi-bound state from which they are emitted. It is therefore possible to describe their motion towards the detectors using such notions of classical mechanics as trajectories and velocities.

Suppose at time $t = 0$, the number of generated resonances was $N(0)$. The counters registering the products of their decays are at a macroscopic distance r . Since the open channels have different thresholds, $E_1 < E_2 < E_3 \dots$, the kinetic energies, $E_{\text{kin}}^{(n)} = E - E_n$, of the decay products are different for the channels. In the channels n and n' the products move towards the counters with the speeds v_n and $v_{n'}$. If $v_n > v_{n'}$, then a pair of simultaneous signals (for the channel-products n and n') means that the particles n were emitted from the resonance at a later time than the particles n' . The time difference is

$$\Delta t = \frac{r}{v_{n'}} - \frac{r}{v_n} = \frac{r(v_n - v_{n'})}{v_n v_{n'}} . \quad (10)$$

If we deal with short-lived resonances, their number (number of the emitters) during the time interval Δt can be significantly reduced. If the particles n' were emitted at $t = 0$, then the intensity of their flux was determined by the number $N(0)$ of the emitters, while for the particles n (which arrive to the detectors simultaneously with n') the number of available emitters was smaller, namely,

$$N(\Delta t) = N(0) \exp\left(-\frac{\Gamma}{\hbar} \Delta t\right) ,$$

and thus their flux was also reduced. Therefore when trying to obtain the branching ratios $\Gamma_n/\Gamma_{n'}$ by comparing the particle-fluxes in the channels n and n' , we should put the corresponding detectors at different distances, r_n and $r_{n'}$, from the emitters. These distances can be found as²

$$r_n = \frac{\hbar k_n}{\mu_n} t , \quad (11)$$

where k_n and μ_n are the wave number and the reduced mass in the channel n , while t is common time of flight (from the emitter to the detector) for all the channels. In a theoretical analysis, this requirement can be fulfilled if we consider the asymptotic behaviour of the resonance wave function (9) at the corresponding distances (11) for different channels. This is what we do in Section 4. The idea of using different asymptotic distances for different channels was taken from Ref. [6].

As was mentioned earlier, in the methods based on the ‘‘golden rule’’ the partial widths exactly sum up to the total width only for narrow resonances. Within these methods, the greater is Γ , the larger is the difference between the left and right hand sides of Eq. (4). The above reasoning may give a possible explanation of such a discrepancy. Indeed, the ratio $N(0)/N(\Delta t)$ exponentially grows with Γ . As we see, this error can be exactly compensated by considering the asymptotics (9) at different distances. But in the integral (6) of the ‘‘golden rule’’ approach there is no possibility for such a correction.

3. Multi-channel wave function

For the sake of clarity, we consider here a simplified problem, namely, the decay of a resonance generated in a two-channel system of spinless particles that interact via non-singular short-range potentials. Its generalization for an arbitrary number of channels and non-zero spins can be done easily. The Coulomb forces can also be included without much difficulties.

² Of course, for fast-moving particles the relativistic kinematics should be used.

The two-channel wave function Ψ has two components, ψ_1 and ψ_2 , that describe the motion in each of the channels. It is convenient to combine them in a column matrix, $\Psi = (\psi_1, \psi_2)^T$, then the Schrödinger equation, $H\Psi = E\Psi$, can be written in a matrix form. When doing the standard partial-wave decomposition of this equation, we treat each partial wave as a separate channel (even if their thresholds are the same). This means that in each of the channels the three-dimensional wave function can be written as

$$\psi_n(E, \vec{r}) = \frac{u_n(E, r)}{r} Y_{\ell_n m_n}(\hat{r}), \quad (12)$$

This gives the following system of matrix radial-equations:

$$\begin{pmatrix} \partial_r^2 + k_1^2 - \frac{\ell_1(\ell_1 + 1)}{r^2} & 0 \\ 0 & \partial_r^2 + k_2^2 - \frac{\ell_2(\ell_2 + 1)}{r^2} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad (13)$$

where the channel momentum $\hbar k_n$ (that determines the relative kinetic energy in the channel n) is given by

$$k_n = \sqrt{\frac{2\mu_n}{\hbar^2}(E - E_n)}, \quad (14)$$

with μ_n and E_n being the reduced mass and the threshold energy in that channel. The matrix V is the angular integral,

$$V_{nn'}(r) = \frac{2\mu_n}{\hbar^2} \int Y_{\ell_n m_n}^*(\hat{r}) U_{nn'}(\vec{r}) Y_{\ell'_n m'_n}(\hat{r}) d\hat{r},$$

of the three-dimensional potential-matrix $U(\vec{r})$.

A system of N differential equations of the second order has $2N$ linearly independent solutions. In our case $N = 2$ and thus Eq. (13) has four independent column-solutions. It is easy to find all of them at large values of r where the (short-range) potential matrix $V(r)$ vanishes. Without the right-hand side, it becomes the Riccati-Bessel equation. Of course the choice of the independent solutions is not unique (like with any set of basis vectors). The most convenient for us is the following choice:

$$\begin{bmatrix} h_{\ell_1}^{(+)}(k_1 r) \\ 0 \end{bmatrix}_1, \quad \begin{bmatrix} 0 \\ h_{\ell_2}^{(+)}(k_2 r) \end{bmatrix}_2, \quad \begin{bmatrix} h_{\ell_1}^{(-)}(k_1 r) \\ 0 \end{bmatrix}_3, \quad \begin{bmatrix} 0 \\ h_{\ell_2}^{(-)}(k_2 r) \end{bmatrix}_4, \quad (15)$$

where $h_\ell^{(\pm)}(z)$ are the Riccati-Hankel functions. These four columns constitute a basis in the space of the solutions of Eq. (13) at large distances, i.e. any other solution is their linear combination. The first two of these columns describe the outgoing spherical waves, while the second pair represents the incoming waves. At large distances, a resonant state may only involve the outgoing waves, i.e. is a combination of the first two columns of (15). Therefore

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \xrightarrow{r \rightarrow \infty} \begin{bmatrix} A_1 h_{\ell_1}^{(+)}(k_1 r) \\ A_2 h_{\ell_2}^{(+)}(k_2 r) \end{bmatrix} \xrightarrow{r \rightarrow \infty} \begin{bmatrix} A_1 (-i)^{\ell_1+1} e^{ik_1 r} \\ A_2 (-i)^{\ell_2+1} e^{ik_2 r} \end{bmatrix}, \quad (16)$$

where A_1 and A_2 are the combination coefficients. The values of them are not arbitrary. They are determined by the dynamics of the system and by a common factor that follows from the chosen normalization. As we will see, the normalization does not play any role in calculating the partial widths, while the relative strengths of the amplitudes A_n can be found using the Jost matrices.

4. Partial widths in the theory

Let us assume that the combination coefficients A_1 and A_2 in Eq. (16) have been found somehow (a way of finding them is suggested in Sec. 5). Using them, we can obtain the fluxes,

$$\vec{\mathcal{J}}_n = \frac{\hbar}{2\mu_n i} \left(\psi_n^* \vec{\nabla} \psi_n - \psi_n \vec{\nabla} \psi_n^* \right),$$

of the decay products in each of the channels at large distances.

We are only interested in the radial component $j_n^{(r)}$ of vector $\vec{\mathcal{J}}_n = \{j_n^{(r)}, j_n^{(\theta)}, j_n^{(\varphi)}\}$, which is normal to a sphere of the radius r . Therefore, out of the three spherical components of the operator $\vec{\nabla} = \{\partial_r, (1/r)\partial_\theta, (1/r \sin \theta)\partial_\varphi\}$, we should use only the radial one. The radial flux for the function (12) with u_n given by Eq. (16), is

$$j_n^{(r)} = \frac{\hbar \text{Re}(k_n)}{\mu_n r^2} |A_n|^2 \exp[-2\text{Im}(k_n)r] Y_{\ell_n m_n}^*(\theta, \varphi) Y_{\ell_n m_n}(\theta, \varphi).$$

Then the total flux in the n -th channel through a distant sphere of the radius r is the surface integral

$$j_n = \int j_n^{(r)} r^2 \sin \theta d\theta d\varphi = \frac{\hbar \text{Re}(k_n)}{\mu_n} |A_n|^2 \exp[-2\text{Im}(k_n)r]. \quad (17)$$

At this point we should recall the discussion given in Section 2 about the choice for each channel a distances r_n which is covered by the decay products during the same time t in all the channels. For a resonance the momentum $\hbar k_n$ is complex. The speed of the products is determined by its real part, and thus

$$r_n = \frac{\hbar \text{Re}(k_n)}{\mu_n} t. \quad (18)$$

At this distance the flux (17) is

$$j_n = \frac{\hbar \text{Re}(k_n)}{\mu_n} |A_n|^2 \exp\left[-2\text{Im}(k_n) \frac{\hbar \text{Re}(k_n)}{\mu_n} t\right] = \frac{\hbar \text{Re}(k_n)}{\mu_n} |A_n|^2 \exp\left(-\frac{\Gamma}{\hbar} t\right), \quad (19)$$

where we used the fact that the total width Γ is expressed via the channel momentum as (this can be seen if the energy (5) is substituted in Eq. (14) instead of E)

$$\Gamma = -\frac{2\hbar^2 \text{Re}(k_n) \text{Im}(k_n)}{\mu_n} \quad \text{for any } n. \quad (20)$$

For all the channels the time-dependent factor in Eq. (19) is the same. Therefore in the ratios $j_n/j_{n'}$ it is always canceled out. Apparently, such a ratio is equal to the ratio of the corresponding decay rates, i.e. it is the branching ratio,

$$\frac{\Gamma_n}{\Gamma_{n'}} = \frac{j_n}{j_{n'}} = \frac{\mu_{n'} \text{Re}(k_n) |A_n|^2}{\mu_n \text{Re}(k_{n'}) |A_{n'}|^2}. \quad (21)$$

The fluxes given by Eq. (19), were originated from the emitter at the same instant of time in all the channels. Therefore the channel flux j_n for a specific n is a fraction f_n of the total flux originated at that time,

$$f_n = \frac{j_n}{j_1 + j_2 + \dots}, \quad (22)$$

and the channel width Γ_n is the same fraction of the total width,

$$\Gamma_n = f_n \Gamma = \frac{\text{Re}(k_n) |A_n|^2 \Gamma}{\sum_{n'=1}^N \frac{\mu_n}{\mu_{n'}} \text{Re}(k_{n'}) |A_{n'}|^2}, \quad (23)$$

where N is the number of open channels. In the wave function (16), all the amplitudes A_n have a common normalization factor, which is canceled out in Eq. (23). We therefore avoid the problem of “proper” normalization of the resonant wave function. What remains is to find a way of obtaining the channel amplitudes A_n . This is done in Section 6.

At this point a question arises about Eq. (8), from which it follows that $\Gamma_n = |\text{Res}(S_{nn}, \mathcal{E})|$. This cannot be completely wrong and thus the residue of the diagonal element of the S -matrix at the resonance pole must somehow be related to the corresponding channel amplitude A_n . And indeed such a relation can be found (see, for example, Ref. [7]),

$$\text{Res}(S_{nn}, \mathcal{E}) = \frac{i\hbar^2 k_n}{\mu_n} (-1)^{\ell_n+1} |A_n|^2, \quad (24)$$

from which we obtain

$$\left| \frac{\text{Res } S_{nn}}{\text{Res } S_{n'n'}} \right| = \frac{\mu_{n'} |k_n| |A_n|^2}{\mu_n |k_{n'}| |A_{n'}|^2}. \quad (25)$$

Comparing this equation with Eq. (21), we see that the above ratio of the residues differs from the ratio of the corresponding partial widths by a kinematical factor,

$$\frac{\Gamma_n}{\Gamma_{n'}} = \left| \frac{\text{Res } S_{nn}}{\text{Res } S_{n'n'}} \cdot \frac{k_{n'} \text{Re}(k_n)}{k_n \text{Re}(k_{n'})} \right|. \quad (26)$$

This kinematical factor, $k_{n'} \text{Re}(k_n) / [k_n \text{Re}(k_{n'})]$, stems from the differences in the time of flight in different channels (discussed in Sec. 2). It becomes noticeable for wide resonances, i.e. when the difference between $|k_n|$ and $\text{Re}(k_n)$ is significant.

Therefore the numerator of Eq. (8) cannot be merely written as $\sqrt{\Gamma_n \Gamma_{n'}}$. It should involve some kinematical factors. These factors may be more complicated than those given in Eq. (26) because in the ratio some of them may cancel. Such kinematical factors must appear also in the “golden rule” formula (6) as a result of a proper normalization of the states $|R\rangle$ and $|\psi_n\rangle$. As we see, the factors become insignificant for narrow resonances. This fact perhaps explains why the “golden rule” produces sufficiently accurate results for them and becomes inaccurate when Γ increases [4, 5].

5. Jost matrices

We assumed that the potential V is not singular. This means that all of its matrix elements are regular or (in the worst case) less singular than $\sim r^{-2}$ at $r = 0$. Therefore the most severe singularity of Eq. (13) is due to the centrifugal terms. However if we multiply the equation by r^2 , none of its coefficients remain singular at $r = 0$. In the theory of differential equations, such points are called the regular singular-points (see, for example, Ref. [8]).

A system of N second-order differential equations has $2N$ linearly independent solutions. At a regular singular-point, half of these solutions are regular while the other half may diverge at such a point. Each solution of Eq. (13) is a column. Therefore there are four linearly independent columns that solve this equation,

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_1, \quad \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_2, \quad \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}_1, \quad \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}_2,$$

where the symbols ϕ and φ are used for the regular and irregular solutions, respectively. These columns can be combined in two square matrices,

$$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}, \quad \varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix},$$

where the second subscript corresponds to the solution number. Since we are only interested in a solution that is relevant to physics, we may forget about the matrix φ and only consider the matrix ϕ whose columns are regular at the point $r = 0$. These columns form a basis in the space of regular solutions. This means that the physical solution u is their linear combination,

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_1 C_1 + \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_2 C_2 = \begin{pmatrix} \phi_{11}C_1 + \phi_{12}C_2 \\ \phi_{21}C_1 + \phi_{22}C_2 \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \quad (27)$$

where the combination coefficients C_1 and C_2 should be chosen in such a way that the function u has a correct asymptotic behaviour (when $r \rightarrow \infty$).

At large distances, where the right-hand side of Eq. (13) vanishes, it still has four linearly independent solutions, namely, the columns (15). When $r \rightarrow \infty$, there is no requirement of regularity. The four columns (15) constitute a full basis there, and any other solution is a linear combination of all of them. In particular, our regular basis ϕ at large distances is also a combination of them,

$$\begin{pmatrix} \phi_{1n} \\ \phi_{2n} \end{pmatrix} \xrightarrow{r \rightarrow \infty} \begin{bmatrix} h_{\ell_1}^{(-)} \\ 0 \end{bmatrix} f_{1n}^{(\text{in})} + \begin{bmatrix} 0 \\ h_{\ell_2}^{(-)} \end{bmatrix} f_{2n}^{(\text{in})} + \begin{bmatrix} h_{\ell_1}^{(+)} \\ 0 \end{bmatrix} f_{1n}^{(\text{out})} + \begin{bmatrix} 0 \\ h_{\ell_2}^{(+)} \end{bmatrix} f_{2n}^{(\text{out})}, \quad n = 1, 2,$$

where the combination coefficients $f_{nn'}^{(\text{in/out})}$ have two subscripts: the first one shows in which channel the column (15) has a non-zero element, and the second subscript indicates which of the columns of the regular matrix is expanded. Combining the regular columns in the square matrix, we have

$$\begin{aligned} \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} &\xrightarrow{r \rightarrow \infty} \begin{bmatrix} h_{\ell_1}^{(-)} f_{11}^{(\text{in})} & h_{\ell_1}^{(-)} f_{12}^{(\text{in})} \\ h_{\ell_2}^{(-)} f_{21}^{(\text{in})} & h_{\ell_2}^{(-)} f_{22}^{(\text{in})} \end{bmatrix} + \begin{bmatrix} h_{\ell_1}^{(+)} f_{11}^{(\text{out})} & h_{\ell_1}^{(+)} f_{12}^{(\text{out})} \\ h_{\ell_2}^{(+)} f_{21}^{(\text{out})} & h_{\ell_2}^{(+)} f_{22}^{(\text{out})} \end{bmatrix} \\ &= \begin{bmatrix} h_{\ell_1}^{(-)} & 0 \\ 0 & h_{\ell_2}^{(-)} \end{bmatrix} \begin{bmatrix} f_{11}^{(\text{in})} & f_{12}^{(\text{in})} \\ f_{21}^{(\text{in})} & f_{22}^{(\text{in})} \end{bmatrix} + \begin{bmatrix} h_{\ell_1}^{(+)} & 0 \\ 0 & h_{\ell_2}^{(+)} \end{bmatrix} \begin{bmatrix} f_{11}^{(\text{out})} & f_{12}^{(\text{out})} \\ f_{21}^{(\text{out})} & f_{22}^{(\text{out})} \end{bmatrix}. \end{aligned}$$

It is not difficult to see that for an arbitrary N , we have the following asymptotic behaviour of the matrix of regular solutions,

$$\phi(E, r) \xrightarrow{r \rightarrow \infty} W^{(\text{in})}(E, r) f^{(\text{in})}(E) + W^{(\text{out})}(E, r) f^{(\text{out})}(E), \quad (28)$$

where $W^{(\text{in/out})}$ are the diagonal matrices

$$W^{(\text{in/out})}(E, r) = \text{diag} \left\{ h_{\ell_1}^{(\mp)}(k_1 r), h_{\ell_2}^{(\mp)}(k_2 r), \dots, h_{\ell_N}^{(\mp)}(k_N r) \right\}, \quad (29)$$

describing the incoming and outgoing multi-channel spherical waves, and the energy-dependent square matrices $f^{(\text{in/out})}$ composed of the combination coefficients, are the multi-channel Jost matrices.

6. Asymptotic behaviour of a resonant solution

Consider the physical solution (27) at large distances,

$$u = \phi C \xrightarrow{r \rightarrow \infty} W^{(\text{in})} f^{(\text{in})} C + W^{(\text{out})} f^{(\text{out})} C, \quad (30)$$

where C is the column-matrix, $C = (C_1, C_2)^T$. By definition, a spectral point (corresponding to either bound or resonant state) is a value of the energy, $E = \mathcal{E}$, for which the wave function has only the outgoing waves in its asymptotic behaviour. From Eq. (30) it follows that this can be achieved if

$$\begin{bmatrix} f_{11}^{(\text{in})} & f_{12}^{(\text{in})} \\ f_{21}^{(\text{in})} & f_{22}^{(\text{in})} \end{bmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0. \quad (31)$$

It is a homogeneous linear system of equations for unknown C_1 and C_2 . It has a non-trivial solution if and only if

$$\det f^{(\text{in})}(\mathcal{E}) = 0. \quad (32)$$

Complex roots of this equation correspond to the resonance energies (5). After finding such a root, we can solve the homogeneous system of linear equations (31) for the unknown coefficients C_1 and C_2 . In the two-channel case this is trivial: C_1 is arbitrary and $C_2 = -C_1 f_{11}^{(\text{in})} / f_{12}^{(\text{in})}$, where C_1 could (in principle) be fixed from the normalization condition. However, as we will see, any common normalization factor is canceled out in the ratio (23).

Now, according to Eq. (30), the resonance-state wave function asymptotically behaves as

$$\begin{aligned} u(\mathcal{E}, r) &\xrightarrow{r \rightarrow \infty} W^{(\text{out})}(\mathcal{E}, r) f^{(\text{out})}(\mathcal{E}) C \\ &= \begin{bmatrix} h_{\ell_1}^{(+)}(k_1 r) & 0 \\ 0 & h_{\ell_2}^{(+)}(k_2 r) \end{bmatrix} \begin{bmatrix} f_{11}^{(\text{out})}(\mathcal{E}) & f_{12}^{(\text{out})}(\mathcal{E}) \\ f_{21}^{(\text{out})}(\mathcal{E}) & f_{22}^{(\text{out})}(\mathcal{E}) \end{bmatrix} \begin{bmatrix} C_1 \\ -\frac{f_{11}^{(\text{in})}(\mathcal{E})}{f_{12}^{(\text{in})}(\mathcal{E})} C_1 \end{bmatrix} = \begin{bmatrix} A_1 h_{\ell_1}^{(+)}(k_1 r) \\ A_2 h_{\ell_2}^{(+)}(k_2 r) \end{bmatrix}, \end{aligned} \quad (33)$$

where A_1 and A_2 ,

$$A_1 = C_1 \left[f_{11}^{(\text{out})} - \frac{f_{11}^{(\text{in})} f_{12}^{(\text{out})}}{f_{12}^{(\text{in})}} \right]_{E=\mathcal{E}}, \quad A_2 = C_1 \left[f_{21}^{(\text{out})} - \frac{f_{11}^{(\text{in})} f_{22}^{(\text{out})}}{f_{12}^{(\text{in})}} \right]_{E=\mathcal{E}}, \quad (34)$$

are the same asymptotic amplitudes of the channels that appear in Eqs. (16) and (23). As is seen, the common factor C_1 can be canceled in Eq. (23). We therefore can put it to unity, $C_1 = 1$.

7. Finding the Jost matrices

The partial widths Γ_n can be found by substituting the channel amplitudes (34) into Eq. (23). Such a calculation requires the knowledge of the Jost matrices $f^{(\text{in})}(\mathcal{E})$ and $f^{(\text{out})}(\mathcal{E})$ at the resonance energy (5). We can obtain these matrices either using the interaction potential (when it is known) or by fitting available experimental data.

7.1. Jost matrices from the potential

If the matrix $V(r)$ of the potential is known, the Jost matrices $f^{(\text{in})}(E)$ and $f^{(\text{out})}(E)$ can be calculated for any complex energy E by a numerical integration of the following system of differential equations (the derivation of these equations and all the details concerning their solutions can be found in Refs. [9–12]):

$$\partial_r F^{(\text{in})} = -\frac{1}{2i} K^{-1} W^{(\text{out})} V \left[W^{(\text{in})} F^{(\text{in})} + W^{(\text{out})} F^{(\text{out})} \right], \quad (35)$$

$$\partial_r F^{(\text{out})} = \frac{1}{2i} K^{-1} W^{(\text{in})} V \left[W^{(\text{in})} F^{(\text{in})} + W^{(\text{out})} F^{(\text{out})} \right], \quad (36)$$

where the diagonal matrix $K = \text{diag} \{k_1, k_2, \dots, k_N\}$, consists of the channel momenta, and the unknown matrices $F^{(\text{in/out})}(E, r)$ asymptotically tend to the Jost matrices,

$$\lim_{r \rightarrow \infty} F^{(\text{in/out})}(E, r) = f^{(\text{in/out})}(E). \quad (37)$$

Starting from the boundary values of these matrices at $r = 0$,

$$F_{nn'}^{(\text{in/out})}(E, 0) = \delta_{nn'}, \quad (38)$$

we can proceed (by numerical integration) to such a distance where the potential vanishes and thus the right-hand sides of Eqs. (35, 36) become zero. Since at that distance the derivatives $\partial_r F^{(\text{in/out})}$ are zero, the limits (37) are reached.

There is a technical difficulty in solving Eqs. (35, 36) for complex energies. Their solutions do not converge to the limits (37) along the real radius r . However this difficulty can be easily overcome if the integration is done along a ray rotated in the complex plane of the coordinate, $r = |r| \exp(i\theta)$, with a large enough angle θ (the details can be found in Refs. [9–12]).

7.2. Jost matrices from the data

The resonance energy, total width, and partial widths can be extracted from any available experimental data by fitting these data with the help of the properly parametrized Jost matrices. In Ref. [13] it was shown that the Jost matrices have the following analytic structure:

$$f_{mn}^{(\text{in})}(E) = \frac{k_n^{\ell_n+1}}{k_m^{\ell_m+1}} a_{mn}(E) - i k_m^{\ell_m} k_n^{\ell_n+1} b_{mn}(E), \quad (39)$$

$$f_{mn}^{(\text{out})}(E) = \frac{k_n^{\ell_n+1}}{k_m^{\ell_m+1}} a_{mn}(E) + i k_m^{\ell_m} k_n^{\ell_n+1} b_{mn}(E), \quad (40)$$

where the energy-dependent matrices $a(E)$ and $b(E)$ are the same for both $f^{(\text{in})}$ and $f^{(\text{out})}$. In the same Ref. [13] it was established that the matrices $a(E)$ and $b(E)$ are single-valued analytic functions of the energy, defined on a single one-layer energy plane, i.e. all the complications stemming from the branch points are isolated in Eqs. (39, 40) via the explicit factors depending on the channel momenta.

Since the matrices $a(E)$ and $b(E)$ are analytic functions of the variable E , they can be expanded in the Taylor series,

$$a(E) = \sum_{n=0}^{\infty} (E - E_0)^n \alpha_n(E_0), \quad b(E) = \sum_{n=0}^{\infty} (E - E_0)^n \beta_n(E_0), \quad (41)$$

near an arbitrary point E_0 within the domain of their analyticity. Here the expansion coefficients α_n and β_n are matrices of the same dimension as a and b . They depend on the choice of the point E_0 .

We can approximate the matrices $a(E)$ and $b(E)$ by a finite number of terms in the expansions (41),

$$a(E) \approx \sum_{n=0}^M (E - E_0)^n \alpha_n(E_0), \quad b(E) \approx \sum_{n=0}^M (E - E_0)^n \beta_n(E_0), \quad (42)$$

and consider the elements of the matrices α_n and β_n as the fitting parameters. The expansion central point E_0 can be chosen on the real axis somewhere in the middle of the available

experimental data (cross sections, analyzing power, etc.). The data can be fitted using the S -matrix

$$S(E) = f^{(\text{out})}(E) \left[f^{(\text{in})}(E) \right]^{-1}, \quad (43)$$

where the Jost matrices depend of the parameters α_n and β_n . After finding the optimal values of these parameters, we can use the same Jost matrices at complex energies within a circle around E_0 . In this way the resonance energy (5) can be located and the channel amplitudes (34) for Eq. (23) can be found.

8. Generalization

In order to make all the derivations clear, we simplified the problem, namely,

- The system under consideration had only two channels ($N = 2$);
- The spins of the decay fragments were zero;
- The interaction potentials were of a short-range type.

The first of these limitations can be dismissed most easily. For an arbitrary N , the dimensions of all the matrices are increased accordingly. When solving Eq. (31) for the unknown coefficients C_1, C_2, \dots, C_N , we can still put $C_1 = 1$ and find all the other of them via the matrix elements of $f^{(\text{in})}(\mathcal{E})$. And finally, Eqs. (34) are to be replaced with more complicated (but still simple) expressions for A_1, A_2, \dots, A_N .

Let the decay fragments have non-zero spins. The total angular momentum J as well as its third component M are conserving. As a result only few partial waves are mixed to each other. If \vec{s} is the total spin of the interacting particles, then $\vec{J} = \vec{\ell} + \vec{s}$, and the maximal number of coupled partial waves is determined by the triangle condition $|J - s| \leq \ell \leq |J + s|$. There is also the parity conservation law, which allows mixing of either only even or only odd angular momenta ℓ .

When generalizing the partial-wave expansion (12), we should use the spin-angular functions (instead of the ordinary spherical harmonics),

$$\mathcal{Y}_{\ell s}^{JM}(\hat{r}) = \sum_{m\mu} C_{\ell m s \mu}^{JM} Y_{\ell m}(\hat{r}) \chi_{s\mu}, \quad (44)$$

where $\chi_{s\mu}$ is the spin function depending on the spin s and its third component μ . For each open channel on the right-hand side of Eq. (1) there is its own set of possible combinations of the pair (ℓs) . Each of these combinations should be considered as a separate channel. For example, if the first two channels are open, namely, $A + B$ and $C + D$, with two and three possible combinations of (ℓs) , respectively,

$$\begin{aligned} n = 1 \quad A + B & : (\ell_1 s_1)_1, (\ell_2 s_2)_1, \\ n = 2 \quad C + D & : (\ell_1 s_1)_2, (\ell_2 s_2)_2, (\ell_3 s_3)_2, \end{aligned}$$

then we have to solve an effective five-channel problem. Apparently, for these effective channels all the formulae given in the previous sections remain the same. After obtaining all five partial widths, we should group them as follows:

$$\begin{aligned} n = 1 \quad A + B & : \Gamma_1 = \Gamma_{(\ell_1 s_1)_1} + \Gamma_{(\ell_2 s_2)_1}, \\ n = 2 \quad C + D & : \Gamma_2 = \Gamma_{(\ell_1 s_1)_2} + \Gamma_{(\ell_2 s_2)_2} + \Gamma_{(\ell_3 s_3)_2}. \end{aligned}$$

The last of our simplifications was the absence of the long-range Coulombic tails of the matrix

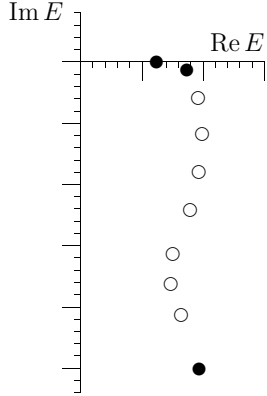


Figure 1. Resonance poles generated by the two-channel model (46, 47). For the poles shown by the filled circles, the resonance parameters are given in Table 1.

Table 1. Parameters of the three resonances shown as the filled circles in Fig. 1.

$\mathcal{E}^{(r)} - \frac{i}{2}\Gamma$	Γ_1	Γ_2	$\frac{\Gamma_1}{\Gamma_2}$	method
	0.037827	0.00049485	76.442	residues
$6.2000 - \frac{i}{2}0.038514$	0.038017	0.00049733	76.442	Masui et al.
	0.038017	0.00049733	76.442	present work
	0.18549	1.5763	0.11768	residues
$8.6519 - \frac{i}{2}1.2992$	0.13679	1.1624	0.11768	Masui et al.
	0.13685	1.1623	0.11773	present work
	58.827	2.4282	24.226	residues
$9.6786 - \frac{i}{2}50.155$	48.166	1.9882	24.226	Masui et al.
	48.214	1.9403	24.848	present work

elements of V . If the decay products have non-zero charges, we should replace the Riccati-Hankel functions $h_\ell^{(\pm)}(kr)$ with the corresponding Coulomb functions,

$$H_\ell^{(\pm)}(\eta, kr) = F_\ell(\eta, kr) \mp iG_\ell(\eta, kr) \xrightarrow{r \rightarrow \infty} (\mp i)^{\ell+1} \exp\{\pm i[kr - \eta \ln(2kr) + \sigma_\ell]\} . \quad (45)$$

It is easy to check that with such a replacement all the formulae for calculating the partial widths remain the same. As far as the differential equations (35,36) and the analytic structure (39,40) of the Jost matrices are concerned, their generalizations for the charged particles can be found in Ref. [12]. In essence, with these generalizations the procedures for finding the Jost matrices either from a potential or by fitting the data remain the same.

9. Numerical example

In order to test the formulae derived in this paper, let us consider a simple two-channel model with the potential

$$U(r) = \begin{pmatrix} -1.0 & -7.5 \\ -7.5 & 7.5 \end{pmatrix} r^2 e^{-r} , \quad (46)$$

given in the units such that $\hbar c = 1$. This is the same potential that is used in the well-known Noro-Taylor model [14]. However, the original model is not sufficiently sensitive to the kinematic factor in Eq. (26). The reason is that the mass is the same in both channels, $\mu_1 = \mu_2 = 1$, and their thresholds are very close to each other, $E_1 = 0$, $E_2 = 0.1$. As a result the time of flight is practically the same in both channels. To avoid this, let us modify the model by using different masses and a larger difference between the thresholds,

$$\mu_1 = 1, \quad \mu_2 = 1.1, \quad E_1 = 0, \quad E_2 = 2. \quad (47)$$

Some resonance poles found within this modified Noro-Taylor model are shown in Fig. 1. For the two narrow and one wide resonance (filled circles in Fig. 1), the partial widths were calculated using four different methods. In the first method it was assumed that Eq. (8) is correct and thus $\Gamma_n = |\text{Res}(S_{nn}, \mathcal{E})|$. The second method was suggested by Masui et al. in Ref. [15], where the authors avoided the problem of normalization in Eq. (8) by taking the ratio of the widths as $\Gamma_1/\Gamma_2 = \text{Res } S_{11}/\text{Res } S_{22}$ and using the condition $\Gamma_1 + \Gamma_2 = \Gamma$. They however did not include the kinematical factor that is present in Eq. (26). Within the third method, the partial widths were calculated using Eq. (26) with this factor included. And lastly, the fourth method was based on Eq. (23) where the asymptotic amplitudes A_n we calculated as is described in Sec. 7.1.

The results of the calculations are given in Table 1. It turned out that the third and the fourth methods produced identical numbers. For each resonance, they therefore are represented in the table by a common line denoted as “present work”.

10. Summary and conclusion

It is shown how the partial widths of a multi-channel resonance can be found using the Jost matrices that determine the asymptotic behaviour of the resonance wave function. After the Jost matrices are obtained, the suggested method does not require any additional integration or differentiation. A normalization of the resonance wave function is not needed. The partial widths are calculated as certain fractions of the total width and thus they exactly sum up to it. These fractions are obtained as the corresponding ratios of the channel-fluxes to the total flux of the decay products. When calculating such ratios, the channel fluxes emitted at the same instant of time are considered.

The proposed method can be used for the analysis of various non-relativistic quantum resonance phenomena in nuclear, atomic, and molecular physics. Depending on the type of the task, the Jost matrices needed for finding the partial widths, can either be calculated from a given potential, or found approximately by fitting available experimental data.

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