Evolution of quark masses and flavour mixings in 5D for an $SU(3)$ gauge group

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Abstract. The evolution of the Cabibbo-Kobayashi-Maskawa matrix elements, the Jarlskog invariant and the quark mixings are derived for the one-loop renormalisation group equations in a five-dimensional model for an SU(3) gauge group compactified on an S_1/Z_2 orbifold. In this work we have assumed that there is a fermion doublet and two singlets located at the fixed points of the extra dimension. This work builds on earlier works of gauge couplings and Yukawa coupling evolutions, which pointed to some interesting phenomenology in this toy model of gauge-Higgs unification.

1. Introduction

The Standard Model (SM) of particle physics is believed to be an effective low energy theory for a number of reasons, where one of these reasons is to try and understand the fermion mass hierarchy and quark mixings [1]. In the SM there is a hierarchy of the quark masses belonging to various generations of the up-type quark masses (m_t, m_c, m_u) and also the down-type quark masses (m_b, m_s, m_d) [2]:

$$m_t \gg m_c \gg m_u, \qquad m_b \gg m_s \gg m_d. \tag{1}$$

In gauge theories, the renormalisable fermion masses come from mass terms such as $\bar{f}Mf$ and also arise from Yukawa terms like $\bar{f}Yf\Phi$. For these Yukawa terms, once the Higgs doublet acquires a vacuum expectation value, all the SM fermions acquire a mass, where this mass is proportional to their Yukawa couplings [3]:

$$Y_{t,c,u} = \frac{m_{t,c,u}}{v} \qquad Y_{b,s,d} = \frac{m_{b,s,d}}{v}. \tag{2}$$

$Y_{t,c,u}$ and $Y_{b,s,d}$ are the nontrivial Yukawa couplings eigenvalues, v is the vacuum expectation value of the Higgs field, where this value can be fixed from the measurement of the W boson mass:

$$v = \frac{2M_W}{g} \simeq 246GeV. \tag{3}$$

In the standard electroweak model with three quark families, the quark sector contains ten free parameters, six quark masses and also four flavour mixing parameters [4]. In order to look into the dynamics of fermion mass and flavour mixing we need to extend the SM. We expected that

any new physics beyond the SM shall appear above the $M_Z \sim 91.2$ GeV scale. In order to build a mass model of quarks at the high energy scale, one can use the Renormalisation Group Equations (RGEs). We need this technique to fill in the space between the predictions of the model at $\mu \gg M_Z$ and the experimental ones at $\mu \leq M_Z$ [5]. We are using these RGEs in order to study the asymptotic behaviour of the Lagrangian parameters, such as Yukawa couplings for both up-type quarks and down-type quarks and also the mixing angles θ_{12}, θ_{13} and θ_{23} [6]. In order to compute the running of quark masses above the M_Z scale we are going to use the quark masses and the mixing parameters, which are obtained at the M_Z scale to determine the Yukawa couplings Y_u and Y_d. After doing this, we need to solve the RGEs of the Yukawa couplings, in order to get the running of the quark masses at any energy scale [5]. In order to diagonalise the quark mass matrices, one can use an unitary matrix as follows [3, 7]:

$$u_L = (U_{u_L})u'_L, \qquad u^c_R = (U_{u^c_R})^\dagger u'_R, \qquad d_L = (U_{d_L})d'_L, \qquad d^c_R = (U_{d^c_R})^\dagger d'_R. \tag{4}$$

However, this will lead to the following:

$$(U_{u^c_R})^\dagger Y_u (U_{u_L}) = \mathrm{diag}(y_u, y_c, y_t), \tag{5}$$

$$(U_{d^c_R})^\dagger Y_d (U_{d_L}) = \mathrm{diag}(y_d, y_s, y_b), \tag{6}$$

or equivalently we can diagonalise the quark mass matrices appearing in the Lagrangian of Yukawa interactions by using the bi-unitary transformation [3, 4]:

$$(U_{u_L})^\dagger M_u (U_{u^c_R}) = \mathrm{diag}(m_u, m_c, m_t), \tag{7}$$

$$(U_{d_L})^\dagger M_d (U_{d^c_R}) = \mathrm{diag}(m_d, m_s, m_b). \tag{8}$$

We use this bi-unitary transformation in order to change all our quark fields from their flavour eigenstate basis to the basis of mass eigenstates [4]. Let us assume that we are working in the basis where the Yukawa couplings for the up-type quark Y_u is diagonal, as appears in Eq. (5), then the mass eigenstates of the down-type quark are connected to their weak eigenstates by the Cabibbo-Kobayashi-Maskawa (CKM) matrix V_{CKM} [5]:

$$V^\dagger_{CKM} Y_d Y^\dagger_d V_{CKM} = \mathrm{diag}(l^2_d, l^2_s, l^2_b). \tag{9}$$

For the other way around, that is, if we are working in the basis in which the Yukawa coupling for the down-type quarks are diagonal, then the mass eigenstates of the up-type quark are given by:

$$V^\dagger_{CKM} Y_u Y^\dagger_u V_{CKM} = \mathrm{diag}(k^2_d, k^2_s, k^2_b). \tag{10}$$

Furthermore, we can build the Yukawa couplings for the down-type quarks from their eigenvalues and also from the CKM matrix [5].

There are many ways to look at the quark mass hierarchy and flavour mixings; we shall investigate an $SU(3)$ gauge group compactified on an S^1/Z_2 orbifold which has size $R^{-1} = 1$ TeV, 5 TeV and 20 TeV. In this paper we assume that the fermion doublet and the two singlet are located at the fixed points of the extra-dimension, the quark masses and the flavour mixings are derived at one-loop level [8].

2. The evolution of CKM matrix in 5 dimension for an SU(3) gauge group

The SM of particle physics has been very successful in describing most of the particle phenomenology known to date [9], but it possesses some problems whose solution could imply physics beyond the SM. The SM is not like QCD and QED, it is the theory which violates parity

(P), time reversal (T) and charge conjugation (C). The C and P separately are not a good symmetry of the SM, but the combination CP, in the case of only one family of matter fields, or even if we have two families, is a good symmetry. Since we have three families in the SM, CP is not a good symmetry. All of the SM Lagrangian is invariant under CP transformations, except the part where the CKM matrix appears.

In order to study the CKM matrix, let us start with $\bar{u}_{iL}\gamma^\mu d_{iL}$ and express it in terms of mass eigenstates.

$$\bar{u}_{iL}\gamma^\mu d_{iL} = \bar{u}'_{hL}(U^{u_L})_{hi}\gamma^\mu(U^{d_L})^\dagger_{ij}d'_{jL} = (U^{u_L})_{hi}(U^{d_L})^\dagger_{ij}\bar{u}'_{hL}\gamma^\mu d'_{jL}, \tag{11}$$

because in the above equation the two matrices are different, when we compute the product of two unitary matrices we still get a unitary matrix. This unitary matrix is called the CKM matrix

$$(U^{u_L})_{hi}(U^{d_L})^\dagger_{ij} \equiv V_{hj}. \tag{12}$$

In order to parameterise the quark sector's flavour mixing we need the CKM matrix [10], and it has 9 parameters. Let us see how the CKM can be parameterised in terms of these 9 parameters[11]:

$$V = \begin{pmatrix} e^{i\tau_1} & 0 & 0 \\ 0 & e^{i\tau_2} & 0 \\ 0 & 0 & e^{i\tau_3} \end{pmatrix} V_{st} \begin{pmatrix} e^{i\sigma_1} & 0 & 0 \\ 0 & e^{i\sigma_2} & 0 \\ 0 & 0 & e^{i\sigma_3} \end{pmatrix}. \tag{13}$$

V_{st} is the standard parametrisation and it is given by:

$$V_{st} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{-i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{14}$$

where $c_{ij} \equiv \cos\theta_{ij}$, $s_{ij} \equiv \sin\theta_{ij}$ [12]. From the standard parameterisation in Eq. (14). The CKM matrix has the following form [13]

$$V_{st} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}. \tag{15}$$

From the experimental point of view we know that $\sin\theta_{13} \ll \sin\theta_{23} \ll \sin\theta_{12} \ll 1$, and we can express this hierarchy using the Wolfenstein parametrisation [14]:

$$\sin\theta_{23} = \frac{|V_{cb}|}{\sqrt{|V_{ud}|^2 + |V_{us}|^2}}, \tag{16}$$

and

$$\sin\theta_{12} = \frac{|V_{us}|}{\sqrt{|V_{ud}|^2 + |V_{us}|^2}}. \tag{17}$$

The RGEs for the CKM matrix beyond the R^{-1} scale is given as follows [15]:

$$16\pi^2\frac{dV_{i\gamma}}{dt} = 12S(\mathrm{t})\left[\sum_{\sigma,j\neq i}\frac{k_i^2+k_j^2}{k_i^2-k_j^2}l_\sigma^2 V_{i\sigma}V^*_{j\sigma}V_{j\gamma} + \sum_{j,\sigma\neq\gamma}\frac{l_\gamma^2+l_\sigma^2}{l_\gamma^2-l_\sigma^2}k_j^2 V^*_{j\sigma}V_{j\gamma}V_{i\sigma}\right], \tag{18}$$

where the energy scale parameter t= $\ln(\mu/M_Z)$ and $S(\mathrm{t}) = M_Z Re^{\mathrm{t}}$. As we mentiond earlier, our renormalisation point is the Z boson mass. Furthermore, we can introduce the Jarlskog

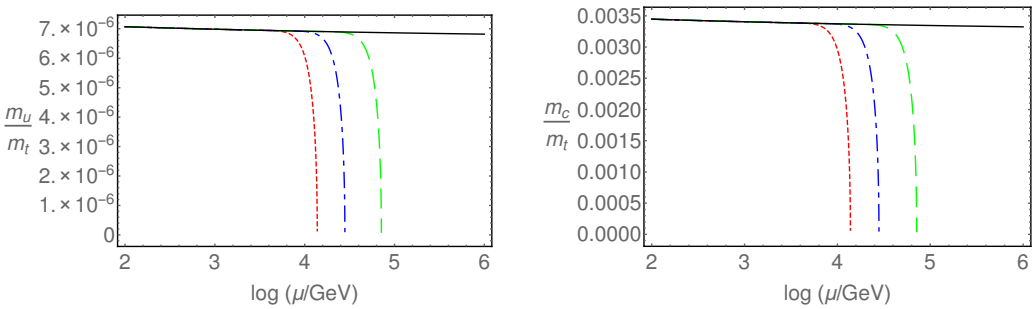

Figure 1. Evolution of the mass ratio for three different values of the compactification radius we have used: 4 TeV (dotted red line), 8 TeV (dot-dashed blue line), 20 TeV (dashed green line); as a function of the scale parameter t. In the left panel is the evolution of the mass ratio m_u/m_t, and the right panel is the evolution of the mass ratio m_c/m_t.

rephasing-invariant parameter J, which is crucial to measuring CP violation, and it is given through the unitarity properties of $V_{\rm CKM}$, as [4]:

$$\mathrm{Im}(V_{k\alpha}V_{l\beta}V^*_{k\beta}V^*_{l\alpha}) = J\sum_{m,\delta}(\varepsilon_{klm}\varepsilon_{\alpha\beta\delta}), \tag{19}$$

where the subscript (k, l or m) runs over the (u, c, t) quarks and the subscript (α, β or δ) runs over the (d, s, b) quarks. In particular, in this paper, we are using the following J to present the CP violation phenomena

$$J = \mathrm{Im}(V_{cs}V_{tb}V^*_{cb}V^*_{tb}). \tag{20}$$

Thus, one can write its square as:

$$J^2 = |V_{tb}|^2|V_{cs}|^2|V_{ts}|^2|V_{cb}|^2 - \frac{1}{4}\left(1-|V_{tb}|^2-|V_{cs}|^2-|V_{ts}|^2-|V_{cb}|^2+|V_{tb}|^2|V_{cs}|^2+|V_{ts}|^2|V_{cb}|^2\right). \tag{21}$$

For completeness, the independent parameters of $V_{\rm CKM}$ are $|V_{ud}|$, $|V_{us}|$, $|V_{cd}|$ and $|V_{cs}|$ and they take the following initial values:

$$|V_{ud}| = 0.9738, \quad |V_{us}| = 0.2196, \quad |V_{cd}| = 0.224, \quad |V_{cs}| = 0.996. \tag{22}$$

We can define the RGE invariant quantity in the hierarchical limit $m_b \gg m_s$ [16]:

$$R_{23} = \sin(2\theta_{23})\sinh\left[\ln\left(\frac{m_b}{m_s}\right)\right]. \quad \Rightarrow R_{23} = \sin\theta_{23}\cos\theta_{23}\left(\frac{m_b}{m_s}\right). \tag{23}$$

3. Numerical Results

In FIG.1 we present the evolution of the mass ratio for the one-loop calculation for three different compactification scales: $R^{-1} = 4$ TeV, 8 TeV and 20 TeV. We expect new physics to come into play when we reach our cut-off. The cut-off for our effective theory when t = 4.1400, 4.4475, 4.8538 for $R^{-1} = 4$ TeV, 8 TeV, 20 TeV respectively. In the left panel we present the evolution of m_u/m_t; in this case one can see that the SM (the black solid line) behaves like λ^8, where $\lambda \sim 0.22$. Through the numerical analysis of the one-loop calculation, we observe that when the fifth dimension contributions switch on, the mass ratio m_u/m_t decreases whenever the energy increases, and this creates a significant change of order of λ^8. In the right panel we are showing the evolution of m_c/m_t, in this case we see that the SM behaves like λ^4, and when the fifth

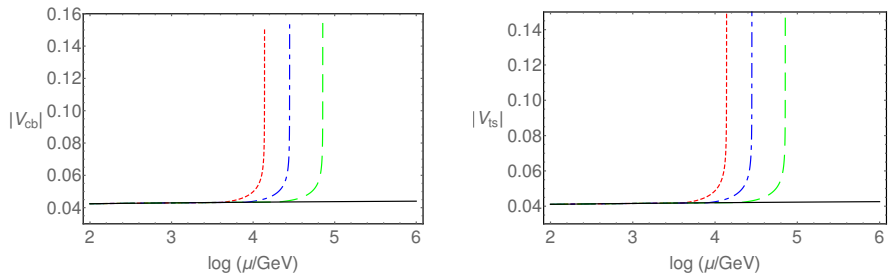

Figure 2. The evolution of the CKM elements for three different values of the compactification radius we have used: 4 TeV (dotted red line), 8 TeV (dot-dashed blue line), 20 TeV (dashed green line); as a function of the scale parameter t. In the left panel is the evolution of the CKM element $|V_{cb}|$, and the right panel is the evolution of CKM element $|V_{ts}|$.

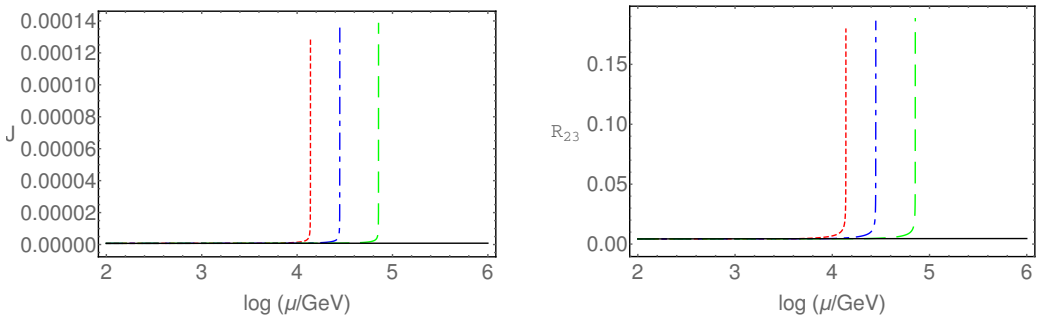

Figure 3. In the left panel is the evolution of the Jarlskog rephasing-invariant parameter; the right panel is the evolution of R_{23}, for three different values of the compactification radius: 4 TeV (dotted red line), 8 TeV (dot-dashed blue line) and 20 TeV (dashed green line); as a function of the scale parameter t.

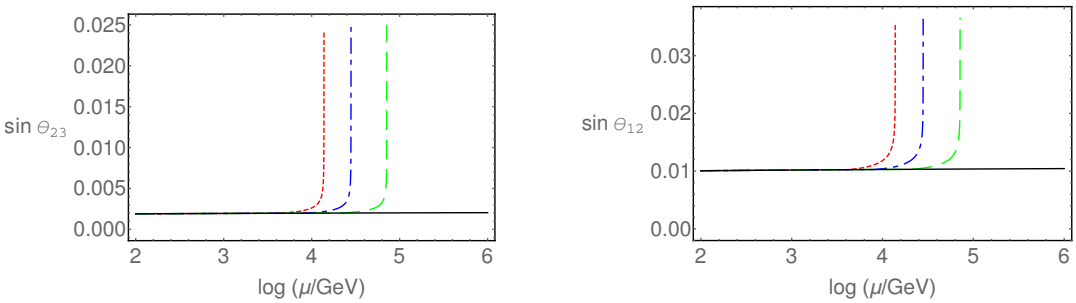

Figure 4. Left panel is the evolution of $\sin\theta_{23}$; the right panel is the evolution of $\sin\theta_{12}$, for three different values of the compactification radius: 4 TeV (dotted red line), 8 TeV (dot-dashed blue line), 20 TeV (dashed green line), as a function of the scale parameter t.

dimension KK-modes become kinematically accessible the mass ratio m_c/m_t decreases with increasing energy, and in this case the change is of the order of λ^4.

In FIG.2 we plot the evolution of the CKM parameters, in the left panel we plot $|V_{cb}|$ and in the right panel $|V_{ts}|$. We see that once the fifth dimension contributions switch on, one can see that there are new contributions coming from the fifth dimension. Accordingly the evolution of the CKM parameters $|V_{cb}|$ and $|V_{ts}|$ are rapidly increasing, this significant increase is of order of λ^2.

In FIG.3, left panel, we plot the Jarlskog invariant parameter. As we mentioned earlier, this gives us a good indication of the amount of CP violation in the quark sector. As can be seen, once the fifth dimension contributions are reached, the value of the Jarlskog invariant increases sharply until we reach the cutoff. In the right panel we present the evolution of the renormalisation invariant R_{23}; this quantity describes the relationship between the mixing angles ($\sin\theta_{23}$ and $\cos\theta_{23}$) and the mass ratio (m_b/m_s) as it appears in Eq. (23). This renormalisation invariant quantity starts to increase rapidly when the fifth dimension contributions switch on. This rapid increase causes increases in the mixing angles, which is suppressed by the mass ratio m_b/m_s; similarly, in FIG.4, in the left panel, we present the evolution of the mixing angle $\sin\theta_{23}$, and in the right panel we plot the evolution of the mixing angle $\sin\theta_{12}$. After, the fifth dimension is switched on, the mixing angles $\sin\theta_{23}$ and $\sin\theta_{12}$ increase rapidly. However, this increase is suppressed by $|V_{cb}|$ and $|V_{us}|$ respectively, as is shown in Eq. (16) and Eqs (17).

4. Conclusion

In conclusion, in this paper we derived the one-loop RGEs in a five-dimensional gauge-Higgs unification model for an $SU(3)$ gauge group by assuming that the fermion doublet and the two singlets are located at the fixed points of the fifth dimension. We test the evolution of the mass ratios m_u/m_t, m_c/m_t, the CKM elements $|V_{cb}|$, $|V_{ts}|$, the Jarlskog rephasing-invariant, R_{23} and the evolution of the mixing angle $\sin\theta_{23}$ and $\sin\theta_{12}$. We observed that when the fifth dimension KK-modes become kinematically accessible all the previous physical observables evolution change rapidly.

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