Computational Physics

Why

- Essential to calculate / visualise theory
- Essential for the analysis of experiments

…. compare theoretical prediction and experimental measurements

On a large scale ….

…… modeling / simulation of complex systems

Trend

- Increasing computing power
- More sophisticated theories
- More complex experiments

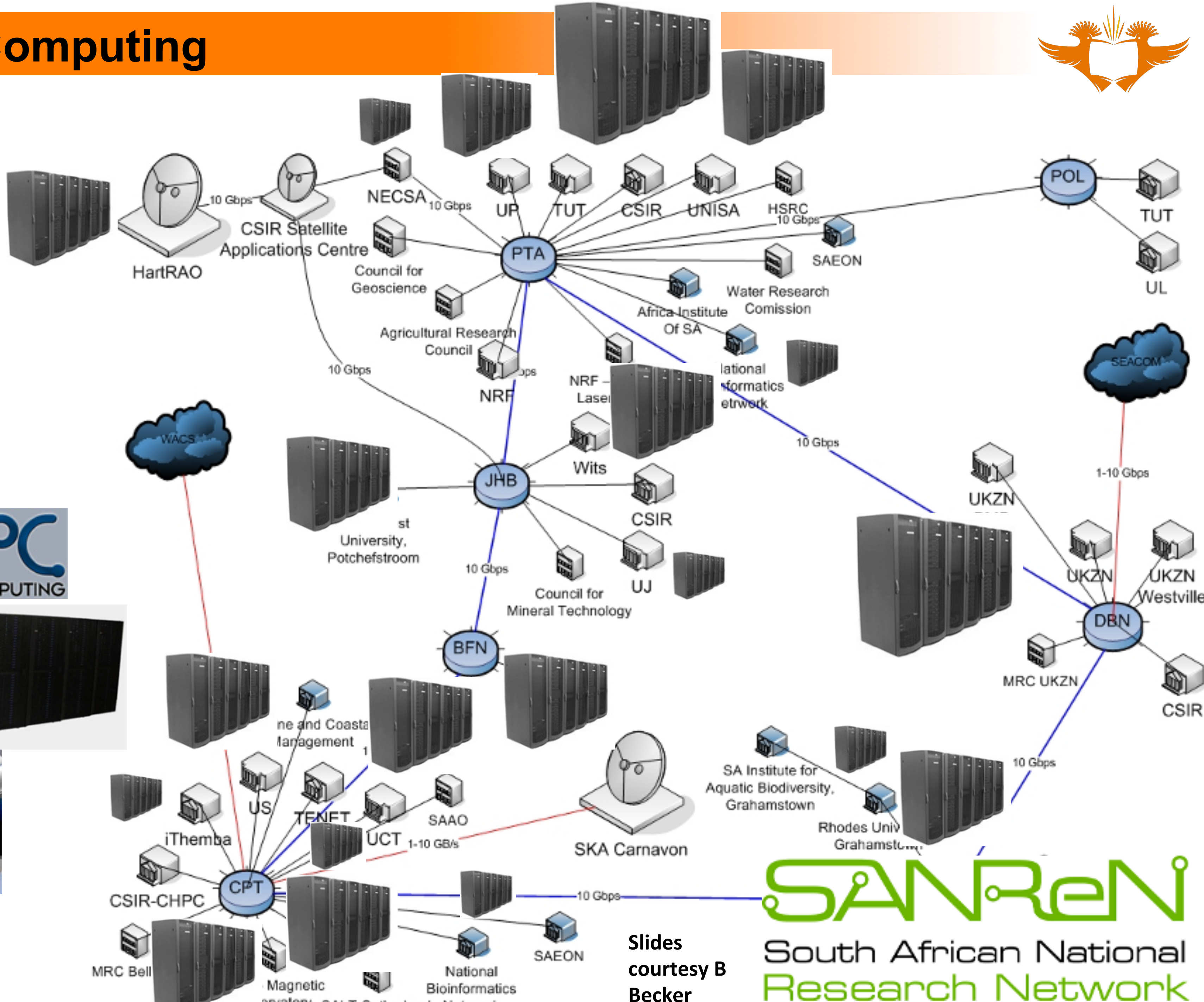
Scalability : Affordable Modern High Performance Computing

Parallel Computing - two relatively affordable

Network of high end PCs and disks
N CPUs, M TB of storage

GPU Computing
100(0)s of CPUs

Grid Computing

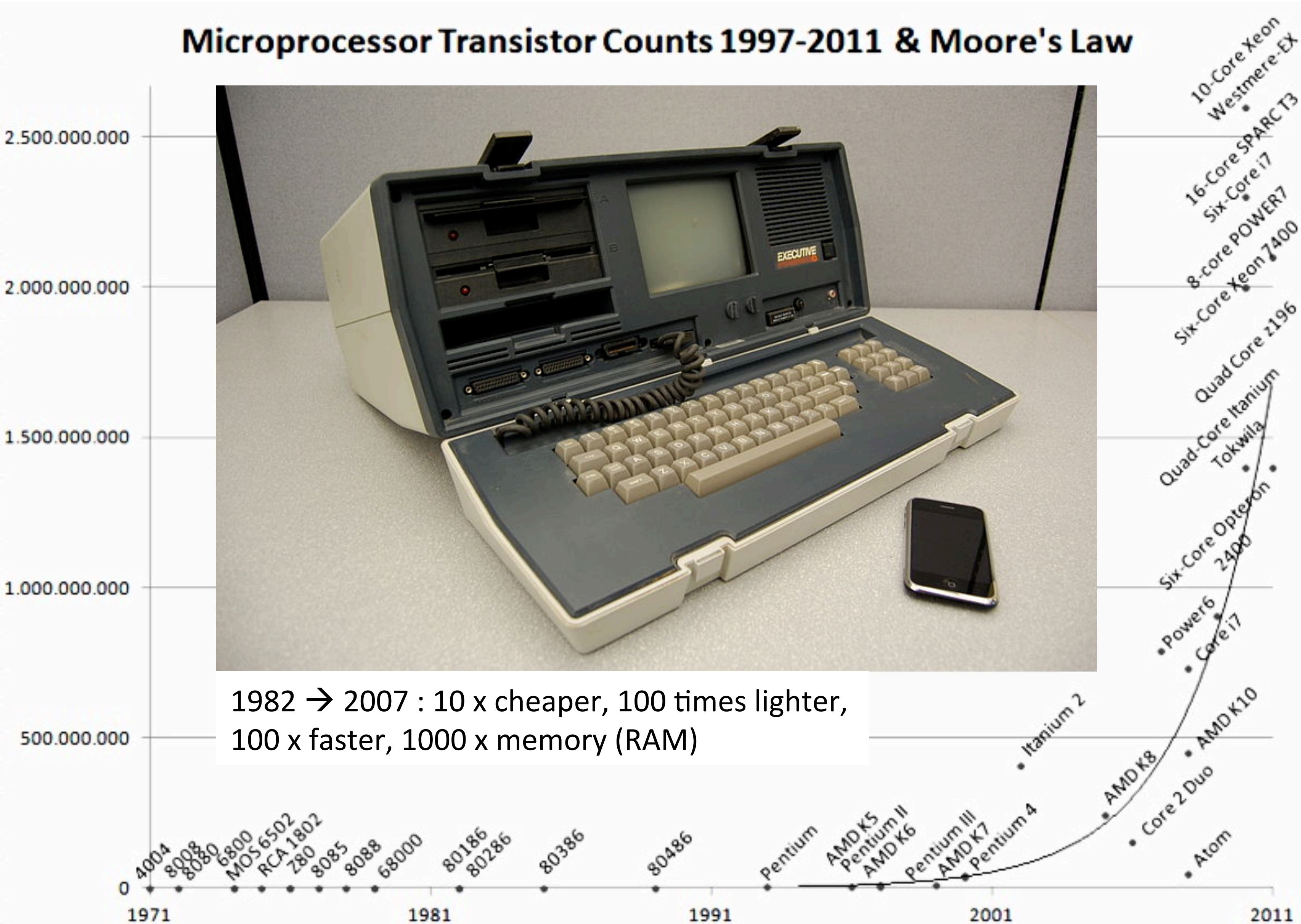


Slides courtesy B Becker

South African National Research Network

2014/11/27

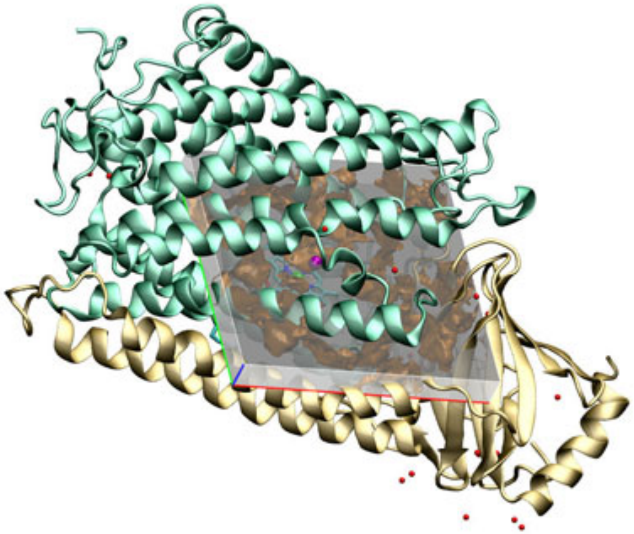
Moore's Law : Computing power doubles every two years

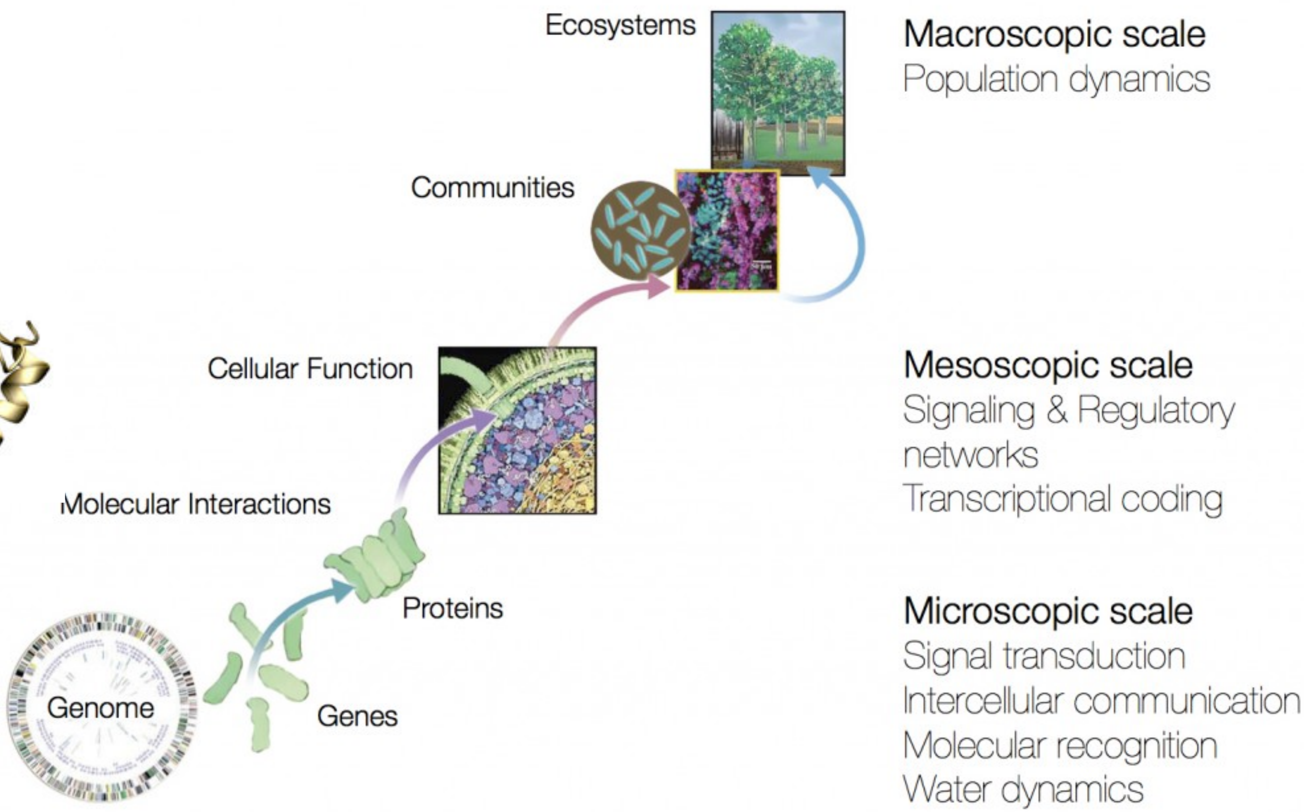


1982 → 2007 : 10 x cheaper, 100 times lighter, 100 x faster, 1000 x memory (RAM)

Examples : Biomedicine

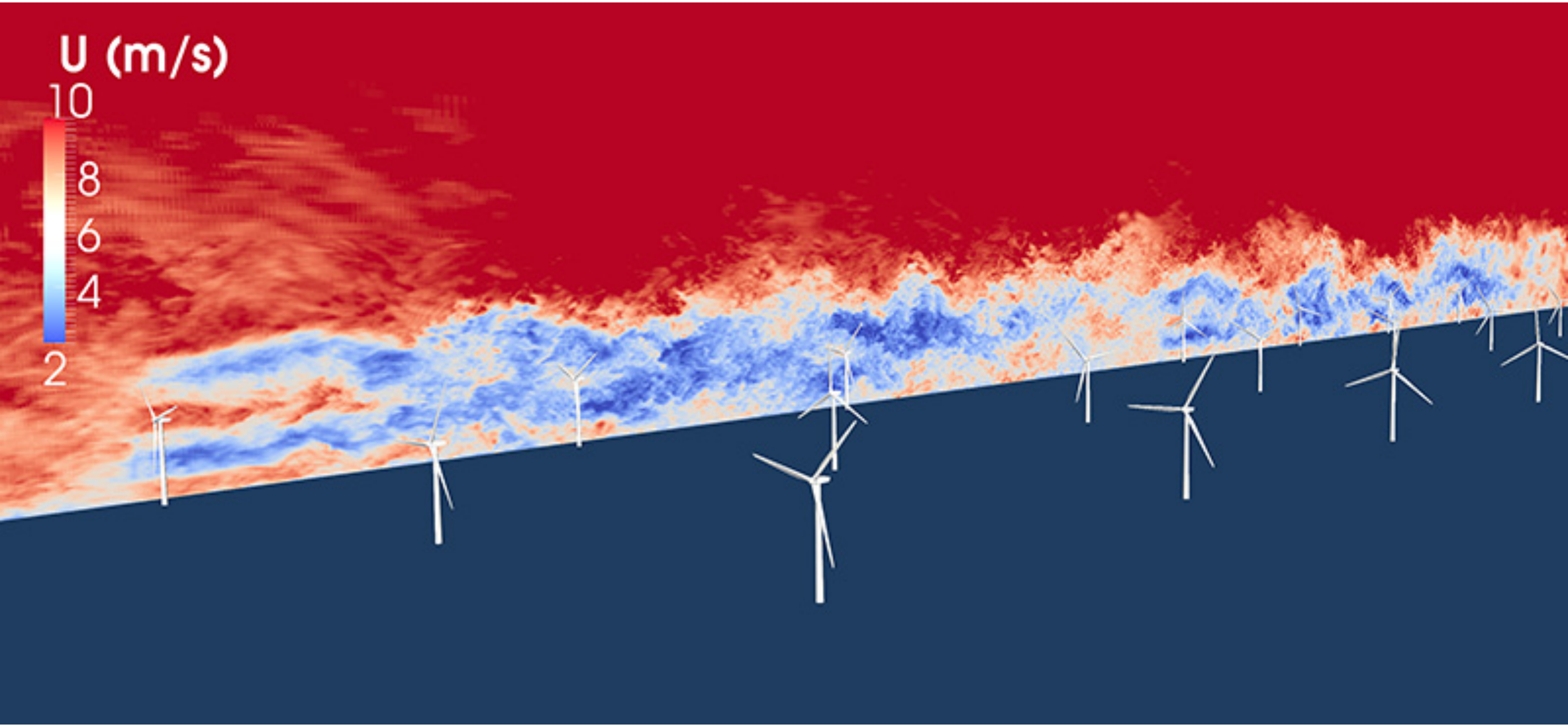
Computational Biology: Multiscale computational modeling of complex biological systems





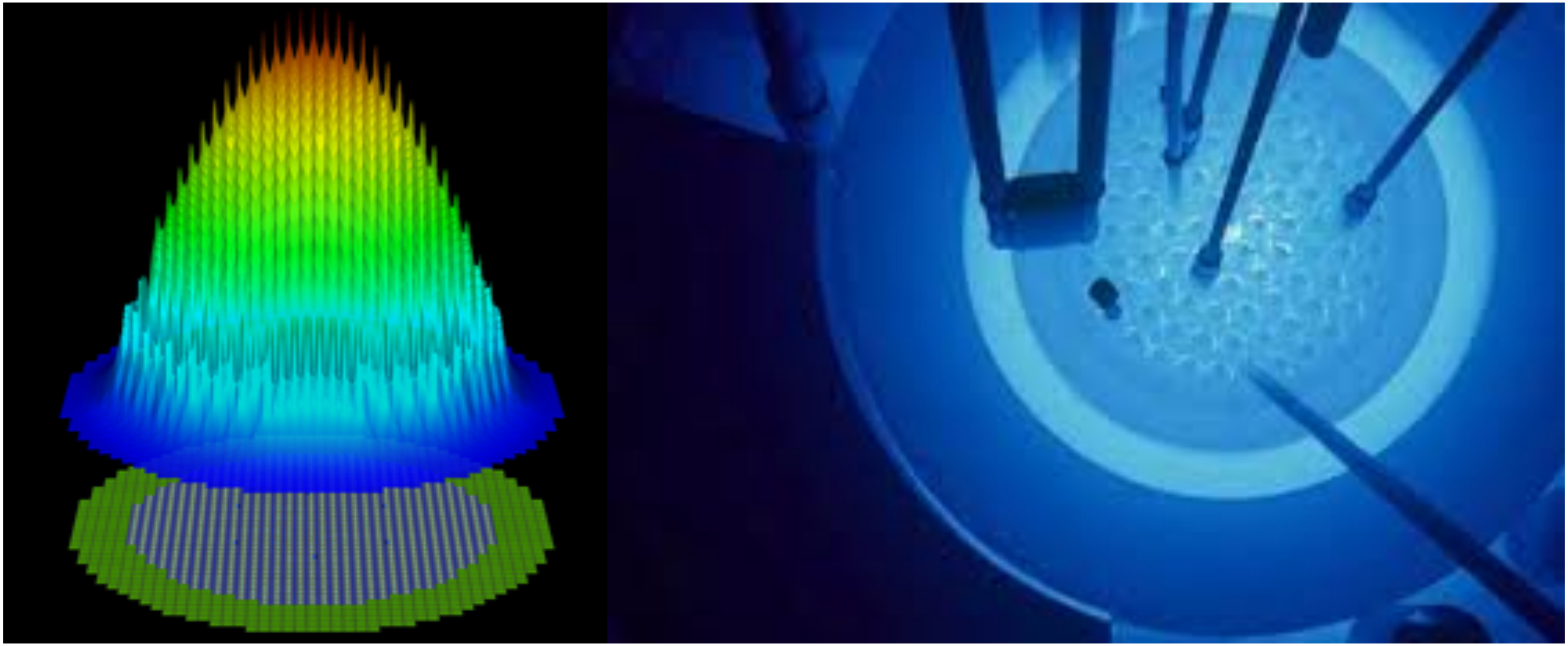
Computer simulations have been employed, for example, to mimic complex neurological processes and reveal the relative strength of human bone structures, development of potential new drugs, etc. This image is obtained from an HPC molecular dynamics simulation of the aa3 enzyme from Paracoccus Denitrificans bacterium and the box representative of the 3D Cartesian grid of points. Image courtesy of Massimiliano Porrini.

Examples : Energy - wind



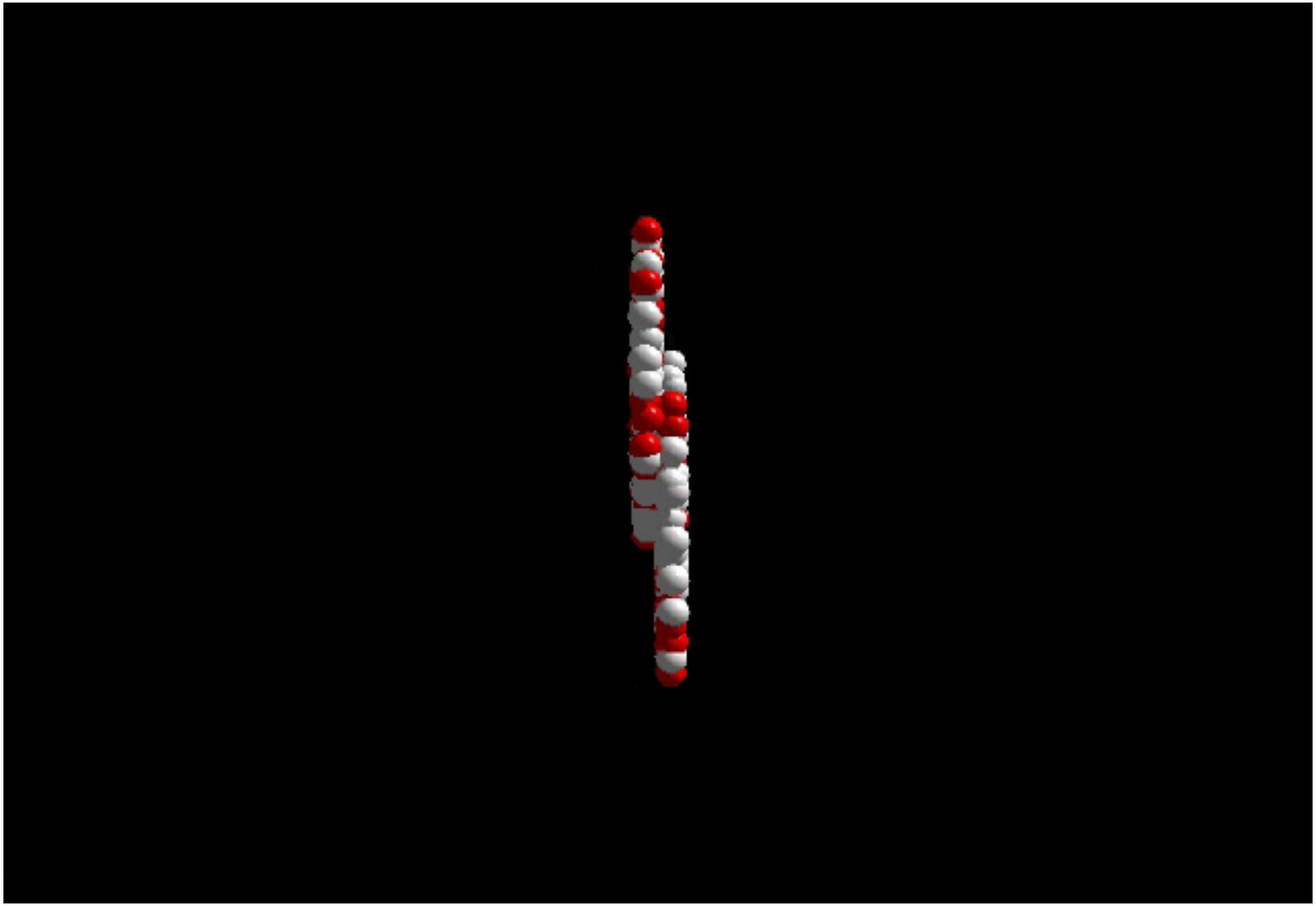
This computer-generated simulation shows the turbulent nature of wind turbine wakes. The simulation helped uncover potential differences in output between downstream 'waked' turbines and upstream turbines.
Simulation by Patrick J. Moriarty and Matthew J. Churchfield, NREL

Examples : Energy - nuclear



An elevation plot of the highest energy neutron flux distributions from an axial slice of a nuclear reactor core is shown superimposed over the same slice of the underlying geometry. This figure shows the rapid spatial variation in the high energy neutron distribution between within each plate along with the more slowly varying, global distribution. UNIC allows researchers to capture both of these effects simultaneously. (Courtesy: Argonne National Lab/Flickr)

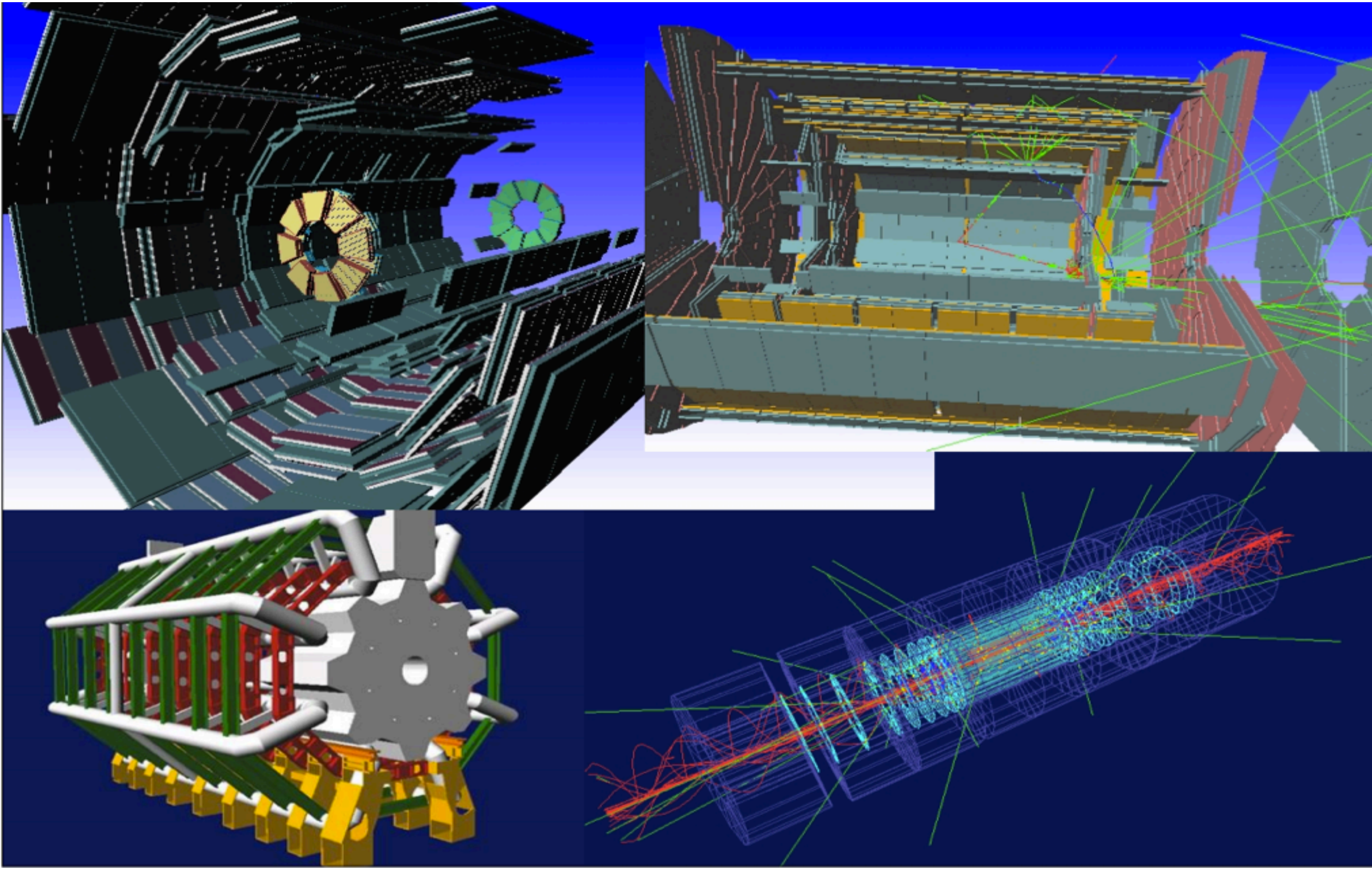
Examples : High energy physics



Extreme relativistic heavy ion collision modelled by Quantum Molecular Dynamics. Protons=red, Neutrons= white, Excited baryons=blue, Mesons=green. Models the creation of dense (excited) hadronic and mesonic matter at high temperatures, the creation and transport of rare particles and strangeness in hadronic matter. and the emission of electromagnetic probes

Examples : High energy physics

GEANT4 Monte Carlo simulation of the ATLAS detector



Computing Platform

- As students, aim ultimately at running on : Clusters / GPUs / GRID
  - ➔ UBUNTU or Mac OS
  - ➔ Windows / Dual boot
  - ➔ Windows / Virtual Machine

Startup Notes

- Simon's Beginners Guide

(http://physics.uj.ac.za/wiki/psi/Computing/Start-upNotesOnProgrammingC)

Tools

- Unix shell environment, (later development environment)
  - ➔ learn about 20 shell commands
- Text Editor (Windows : notepad++, Unix : gedit, (x)emacs, vi(m)….)
  - ➔ learn to create and edit "ascii" files
- C++ Programming Language (http://www.cplusplus.com/doc/tutorial/)
  - ➔ learn to create / edit / debug code
- Compiler, Linker
  - ➔ learn about "make" files
- Numerical Methods (http://apps.nrbook.com/c/index.html)
  - ➔ learn to libraries as white/grey boxes

Practice …..

Create, Edit Text file

- Use the examples at

(http://physics.uj.ac.za/wiki/psi/Computing/Start-upNotesOnProgrammingC)

and

(http://www.cplusplus.com/doc/tutorial/)

Learn C++ by doing tutorials

- At least up to the beginning of "Object Oriented Programming"

(http://www.cplusplus.com/doc/tutorial/)

Play with the White Dwarf Application

- Will be on the HDM2013 www-site

HDM 2014 : Computational Methods - ODEs

SH Connell

Ordinary Differential Equations

See Numerical Recipes - The Art of Scientific Computing by Press et al and Computational Physics by S Koonin and D Meredith.

1 General 2nd order D.E.

Consider the following 2nd order differential equation :

$$\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} = r(x)$$

It can be rewritten as two coupled first order differential equations :

$$\begin{aligned} \tfrac{dy}{dx} &= Z(x) \\ \tfrac{dz}{dx} &= r(x) - q(x)Z(x) \end{aligned}$$

Of course, don't forget the boundary conditions.
A well known example is Newton's Second Law of Motion, which can also be expressed by Hamilton's equations :

$$\text{Newton II} \quad \Rightarrow \quad m\tfrac{d^2z}{dt^2} = F(z) \quad \Rightarrow \quad \begin{cases} \tfrac{dZ}{dt} = p/m \\ \tfrac{dP}{dt} = F(Z) \end{cases}$$
$$\Rightarrow \quad \text{Hamilton eqns.}$$

If the original 2nd order D.E. was in 3-dimensions, then there would be 6 coupled 1st order D.E.'s.
In general, we will consider that we have n coupled 1st order D.E's to solve.

$$\frac{d\tilde{y}}{dx} = \tilde{f}(x, \tilde{y})$$

$\hookrightarrow$ solve $\frac{dy}{dx} = f(x, y)$ and generalise with matrix methods.

We will study ....
$\hookrightarrow$ initial value problems (now)
boundary value problems (later)

2 Euler's Method

- conceptually important
- not recommended - inaccurate - unstable

We have the D.E. and initial condition :

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) \sim c$$

which is discretised on a step size of h. The derivative is then replaced simply by its approximation being the slope over that step size:

$$\frac{y_{n+1} - y_n}{h} + O(h^2) = f(x_n, y_n).$$

We will call this the Euler step (see the Runge Kutta methods discussed later).

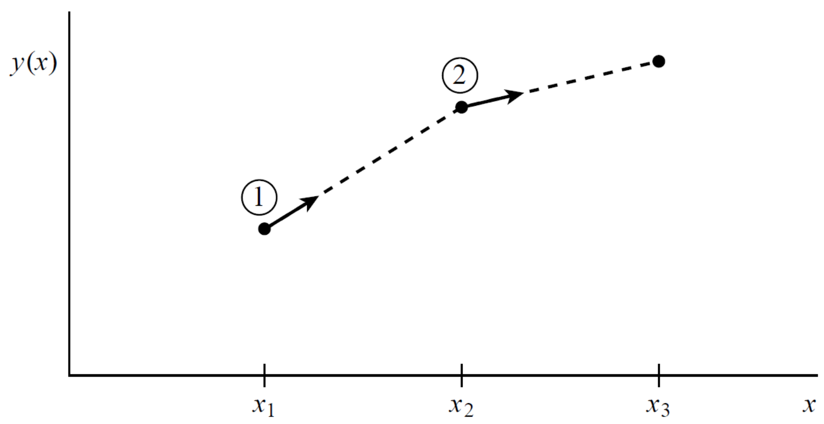


Figure 1: Eulers method. In this simplest (and least accurate) method for integrating an ODE, the derivative at the starting point of each interval is extrapolated to find the next function value. The method has first-order accuracy. (Numerical Recipes Figure 16.1.1.)

This leads to the recursion relation

$$\Rightarrow \qquad y_{n+1} = y_n + hf(x_n, y_n) + 0(h^2).$$

Since

$$h = [b - a]/n$$

the nett error is accumulated from the error at each step $\sim nO(h^2) \sim \frac{1}{h}O(h^2) \sim O(h)$.

3 Runge-Kutte Methods

These are single step methods, there are a variety of algorithms, of the Taylor Series type.

$$\begin{aligned} \tfrac{dy}{dx} = f(x, y(x)) &\rightarrow y_{n+1}(x) = y_n(x) + \textstyle\int_{xn}^{x_{n+1}} f(x, y(x))dx \\ f(x) &\rightarrow \text{Taylor Exp} \end{aligned}$$

2nd Order R-K

For example, perform the Taylor Expansion to 2nd Order, about the midpoint of the interval, once with a half-step to the beginning of the interval, once with a half-step to the end of the interval.

$$\begin{aligned} y(x_n) &= y(x_{n+\frac{1}{2}}) - \frac{h}{2}\frac{dy(x)}{dx}\Big|_{x_{n+\frac{1}{2}}} + \frac{h^2}{8}\frac{d^2y(x)}{dx^2})\Big|_{x_{n+\frac{1}{2}}} + \ldots \\ y(x_{n+1}) &= y(x_{n+\frac{1}{2}}) + \frac{h}{2}\frac{dy(x)}{dx}\Big|_{x_{n+\frac{1}{2}}} + \frac{h^2}{8}\frac{d^2y(x)}{dx^2})\Big|_{x_{n+\frac{1}{2}}} + \ldots \end{aligned}$$

Rewriting ... and introducing an obvious notation ...

$$\begin{aligned} y_n &= y_{n+\frac{1}{2}} - \frac{h}{2}f(x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}) + \frac{h^2}{8}f'(x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}) + \ldots \\ y_{n+1} &= y_{n+\frac{1}{2}} + \frac{h}{2}f(x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}) + \frac{h^2}{8}f'(x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}) + \ldots \end{aligned}$$

Subtracting ....

$$y_{n+1} = y_n + hf(x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}) + O(h^3)$$

We have achieved a higher order by using the symmetric expansion about the interval midpoint. However, we do not know how to evaluate $y_{n+\frac{1}{2}}$, so we approximate this with a half Euler step.

$$y_{n+\frac{1}{2}} = y_n + \frac{1}{2}k, \qquad k = \text{Euler difference} = hf(x_n, y_n)$$

We finally arrive at the 2nd Order Runge-Kutta method:

$$\begin{aligned} y_{n+1} &= y_n + hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k) + 0(h^3) \\ k &= hf(x_n, y_n) \end{aligned}$$

This has two function evaluations per step, but is quite accurate and stable.

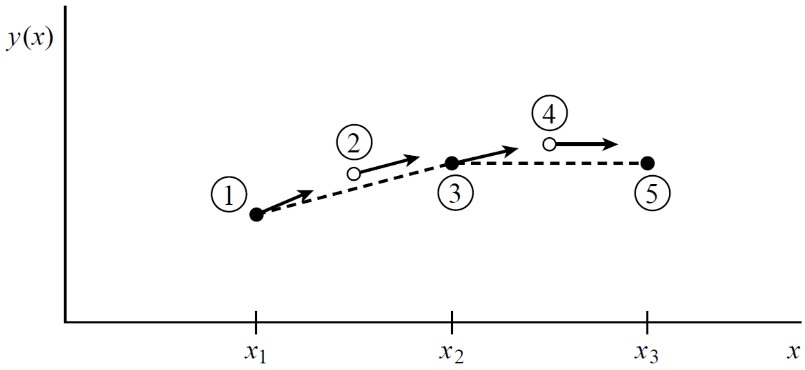


Figure 2: The 2nd Order R-K, or the Midpoint method. Second-order accuracy is obtained by using the initial derivative each step to find a point halfway across the interval, then using the midpoint derivative across the full width of the interval. In the figure, filled dots represent final function values, while open dots represent function values that are discarded once their derivatives have been calculated and used. (Numerical Recipes Figure 16.1.2.)

4th Order R-K

The principles of the 2nd Order R-K (using well chosen implementations of the Taylor series expansion to develop R-K steps where higher orders of error terms are canceled out) can be carried further. For example, in the 4th Order R-K, the following set of steps are implemented. The various steps are self-explanatory :

$$
\begin{aligned}
k_1 &= hf(x_n y_n) \\
k_2 &= hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1) \\
k_3 &= hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_2) \\
k_4 &= hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_3) \\
y_{n+1} &= y_n + \tfrac{k_1}{6} + \tfrac{k_2}{3} + \tfrac{k_3}{3} + \tfrac{k_4}{6} + O(h5)
\end{aligned}
$$

These steps combine to cancel errors to $O(h^5)$

A R-K based general O.D.E. Integrator

Additional features in the numerical integration of an O.D.E. is the estimation of the error at each step and the implementation of an adaptive step size (based on this error estimation). An efficient implementation of this, based on a 5th Order R-K (with the error for each step estimated by a 4th Order R-K based on the same function evaluations) is presented in the Numerical Recipes book in Chapter 16.2.

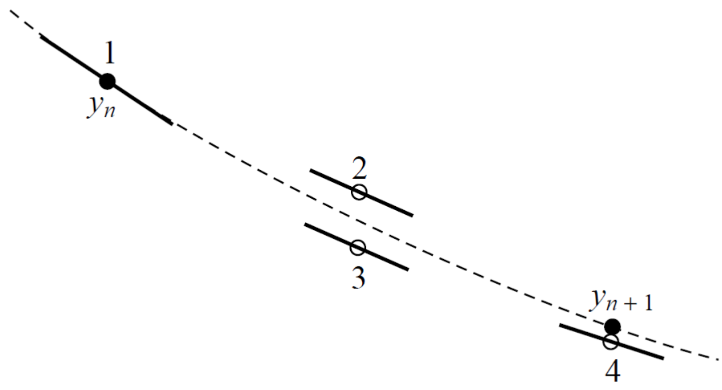


Figure 3: Fourth-order Runge-Kutta method. In each step the derivative is evaluated four times: once at the initial point, twice at trial midpoints, and once at a trial endpoint. From these derivatives the final function value (shown as a filled dot) is calculated. (Numerical Recipes Figure 16.1.3.)

void odeint(float ystart[], int nvar, float x1, float x2, float eps, float h1, float hmin, int *nok, int *nbad, void (*derivs)(float, float [], float []), void (*rkqs)(float [], float [], int, float *, float, float, float [], float *, float *, void (*)(float, float [], float [])))

(See this text for further details)

4 Example : The Structure of White Dwarfs

A white dwarf is a possible destiny of certain stars, as given by theories on nucleo-syntheses and stellar-evolution. It is a relatively cold star where the nucleo-synthesis process is completed. It consists of a plasma of the heavy stable nuclei and their electrons.

In the simple model of this example, the white dwarf can be considered spherically symmetric, non-rotating and magnetic fields are neglected. The structure is considered to be dominantly determined by the hydrostatic equilibrium between gravitational pressure seeking to compress the stellar material and Pauli pressure which resists this.

In this example, we will further imagine that the nucleo-synthesis process has run to completion leaving the star dominantly composed of a single stable nucleus type terminating the fusion cycle (the type of which depends on the mass of the star). For example, ^{12}C or ^{56}Fe.

The electrons are modeled as a $T = 0$ free Fermi gas, and they dominate the Pauli pressure term.

Under these conditions, it can be shown that the white dwarf stellar structure is contained in two coupled first order differential equations for the radial mass and density distribution of the .

Considering firstly the radial mass distribution :
The mass of a sub-sphere to radius $r < R_0$ is

$$m(r) = 4\pi \int_0^r \rho(r') r'^2 dr', \tag{1}$$

from which we get the relationship of m and r by differentiating

$$\frac{dm}{dr} = 4\pi r^2 \rho(r). \tag{2}$$

Considering next the radial density distribution :
As in electrostatics, the gravitational force per unit volume at a given radial distance r from the centre of the star is dependent on the amount of matter enclosed by a sphere of that same radius.

$$\frac{dP}{dr} = \frac{dF(r)}{dV} = -\frac{Gm(r)}{r^2}\rho(r) \tag{3}$$

We seek $\frac{d\rho}{dr}$. Considering the chain rule $\frac{dP}{dr} = \frac{d\rho}{dr}\frac{dP}{d\rho}$, we write :

$$\frac{d\rho}{dr} = -\left(\frac{dP}{d\rho}\right)^{-1} \frac{Gm(r)\rho(r)}{r^2} \tag{4}$$

We just require $\frac{dp}{d\rho}$, the equation of state for the white dwarf.
The initial condition $\rho(r = 0) = \rho_c$ (and obviously $m(r = 0) = 0$) will determine the final mass M and

radius R of the star, $M = m(R)$, by integration of the two coupled first order differential equations for $m(r)$ and $\rho(r)$.

The equation of state is in fact the Pauli pressure term mentioned above.
To determine the Equation of State, we have assumed,

- The star consists of heavy nuclei and e^-.
- The nuclei contribute to the mass, but not to the pressure.
- The e^- contribute to the pressure, but not to the mass.
- The density is so high that e^- are free.

We begin with the energy density of the electrons in units where $\hbar = c = 1$. The electron density is

$$n = Y_e \frac{\rho}{m_n} \tag{5}$$

where Y_e is the number of electrons per nucleon and m_n is the nucleus mass. For a

$$^{56}\text{Fe star} \qquad Y_e = \frac{26}{56} \tag{6}$$

$$^{12}\text{C star} \qquad Y_e = \frac{1}{2} \tag{7}$$

The number of electrons = number of protons for charge neutrality.
If N is the number of electrons, the for a free Fermi gas we have :

$$N = 2V \int_0^{pf} \frac{d^3p}{(2\pi)^3} \tag{8}$$

where the Fermi momentum p_f is

$$p_f = (3\pi^2 n)^{\frac{1}{3}} \qquad \text{with} \qquad n = N/V \tag{9}$$

The energy density is then

$$\frac{E}{V} = 2 \int_0^{p_f} \frac{d^3p}{(2\pi)^3} (p^2 + m_e^2)^{1/2}. \tag{10}$$

Upon integration of equation **??**, we have

$$\frac{E}{V} = n_0 m_e x^3 \epsilon(x) \tag{11}$$

where

$$\epsilon(x) = \frac{3}{8x^3} \left[x(1+2x^2)(1+x^2)^{1/2} - \log[x + (1+x^2)^{1/2}] \right] \tag{12}$$

and

$$x = \frac{p_f}{m_e} = \left(\frac{n}{n_0}\right)^{1/3} = \left(\frac{\rho}{\rho_0}\right)^{1/3} \tag{13}$$

with

$$n_0 = \frac{m_e^3}{3\pi^2} = 5.89 \times 10^{29}\text{cm}^{-3} \qquad \text{and} \qquad \rho_0 = \frac{m_n n_0}{Y_e} = 9.79 \times 10^5 Y_e^{-1} \text{gm cm}^{-3}. \tag{14}$$

Thus x is the electron density in units of n_0, chosen so that $x = 1$ corresponds to the Fermi momentum p_f being equal to the electron mass m_e (see equations **??** and **??**). Noting that $n = N/V$, we see that

$$\frac{\partial x}{\partial V} = -\frac{x}{3V} \tag{15}$$

According to thermodynamics, the pressure is the change in energy with volume, so that

$$\begin{aligned} P &= -\frac{\partial E}{\partial V} \\ &= -\frac{\partial E}{\partial x}\frac{\partial x}{\partial V} \\ &= \tfrac{1}{3} n_0 m_e x^4 \frac{d\epsilon(x)}{dx} \end{aligned} \tag{16}$$

Because the electron density scales with the density, $x \equiv x(\rho)$, equation **??** is indeed the equation of state. We can now evaluate

$$\begin{aligned} \frac{dP(\rho)}{d\rho} &= \frac{dP(x)}{dx}\frac{dx(\rho)}{d\rho} \\ &= Y_e \cdot \frac{m_e}{m_n} \cdot \frac{x^2}{3(1+x^2)^{1/2}} \end{aligned} \tag{17}$$

The two coupled first order differential equations then become :

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r) \tag{18}$$

and

$$\frac{d\rho(r)}{dr} = -\frac{G}{Y_e(m_e/m_n)} \cdot \frac{m(r)}{r^2}\frac{\rho(r)}{\gamma(x(\rho))}$$

where

$$\gamma(x) = \frac{x^2}{3(1+x^2)^{1/2}}. \tag{19}$$

Converting to dimensionless variables $\bar{r}$, $\bar{\rho}$ and $\bar{m}$ for the stellar radius, density and mass,

$$\frac{d\bar{m}}{d\bar{r}} = \bar{r}^2\bar{\rho} \tag{20}$$

and

$$\frac{d\bar{\rho}}{d\bar{r}} = -\frac{\bar{m}\bar{\rho}}{\gamma\bar{r}^2}$$

where

$$r = R_0 \bar{r}, \quad \rho = \rho_0 \bar{\rho}, \quad m = M_0 \bar{m}, \quad x = \bar{\rho}^{1/3} \tag{21}$$

and the scale factors have been chosen to achieve unit coefficients in equation **??**,

$$\begin{aligned}
R_0 &= \left[\frac{Y_e(m_e/m_n)}{4\pi G \rho_0}\right]^{1/2} = 7.72 \times 10^8 Y_e \text{cm}, \\
M_0 &= 4\pi R_0^3 \rho_0 = 5.67 \times 10^{33} Y_e^2 \text{gm} \\
\rho_0 &= \frac{m_n n_0}{Y_e} = 9.79 \times 10^5 Y_e^{-1} \text{gm}\,\text{cm}^{-3}.
\end{aligned} \tag{22}$$

Note that solutions for different values of Y_e may all be scaled from $Y_e = 1$.

Development of the Solution

The solutions to the questions below are available to registered course participants in the protected pages for the course.

a) As a partial approximate analytic solution, find the leading terms in the $\epsilon(x)$ and $\gamma(x)$ expressions for the non-relativistic ($x \ll 1$) limit and discuss the plausibility of the results. Using the result for small $\bar{\rho}$, i.e. finding an analytic approximation near the surface with $\bar{m}$ and $\bar{r}$ finite, and working from equations **??**, show the white dwarf has a well defined surface and find the functional dependence of $\bar{\rho}$ near the surface.

b) Solve numerically for $\bar{\rho}(\bar{r})$ and $\bar{m}(\bar{r})$. Mention any steps you could take, of a numerical nature, that would generate confidence in the numerical result. Calculate the stellar total mass and radius for centre densities $\bar{\rho}_c = \bar{\rho}(0)$ in the range 10^{-1} to 10^6. Does the behaviour of the solution correspond to your approximate analytical solution ? (explain).

c) Deduce the composition of Sirius B (dominantly iron or carbon). The measured parameters for Sirius B are $R = 0.0074(6) R_\odot$, $M = 1.05(3) M_\odot$. The values have been given in solar units, where

$$\begin{aligned}
R_\odot &= 6.95 \times 10^{10} \text{cm}, \\
M_\odot &= 1.98 \times 10^{33} \text{gm}.
\end{aligned} \tag{23}$$

d) Investigate the behaviour of the mass and radius of the white dwarf for large central densities. You will find the mass approaches a limit and the radius collapses. Deduce the Chandrasekhar limit for the mass and central density of the white dwarf star. The destiny of such a large star would be a neutron star.

e) Explain the previous result using a simple model where the density profile is constant $d\bar{\rho}/d\bar{r} = M/V$. Calculate the total energy $U + W$ of the star where

$$U = \int_0^R \left(\frac{E}{V}\right) 4\pi r^2 dr \tag{24}$$

arises from the repulsive Pauli pressure and

$$W = -\int_0^R \frac{Gm(r)}{r^2} \rho(r) 4\pi r^2 dr \tag{25}$$

arises from the attractive gravitational potential. Show that for a given total mass M and at large densities (extreme relativistic limit) then both terms have a $1/R$ dependence and predict the Chandrasekhar collapse for large enough M.

White Dwarf : Solution

a) i) Non-relativistic limit.

$p_F << m_e \Rightarrow x << 1$

Note

$$\frac{E}{V} = 2\int_0^{p_F} \frac{d^3p}{(2\pi)^3}(p^2+m^2)^{1/2} \simeq \frac{2m}{(2\pi)^3}\int_0^{p_F} d^3p \sim p_F^3$$

$$\therefore \frac{E}{V} \sim n \;(\propto \rho)$$

White Dwarf : Solution

now :

$$\epsilon'(x) = \frac{3}{8x^3}\left[x(1+2x^2)\left(1+\frac{x^2}{2}+\frac{x^4}{8}+\cdots\right)-\log\left(x+\left(1+\frac{x^2}{2}+\frac{x^4}{8}+\cdots\right)\right)\right]$$

$$= \frac{3}{8x^3}\left[x+\frac{5}{2}x^3+O(x^5)-\log\left(1+\left(x+\frac{x^2}{2}+\frac{x^4}{8}\right)+\cdots\right)\right]$$

$$\simeq \frac{3}{8x^3}\left[x+\frac{5}{2}x^3+O(x^5)-\left(\left(x+\frac{x^2}{2}+\frac{x^4}{8}\right)-\frac{(\quad)^2}{2}+\frac{(\quad)^3}{3}+O(x\right.\right.$$

$$= \frac{3}{8x^3}\left[\frac{5}{2}x^3-\frac{x^3}{3}+O(x^4)\right] = k + O(x)$$

$$\therefore \frac{E}{V} = n_0 m x^3 \epsilon(x) \sim x^3 + O(x^4)$$

$$\sim n \; (\text{or } \rho) \quad \text{for } x \ll 1$$

$\therefore$ consistent

$$\gamma(x) = \frac{x^2}{3(1+x^2)^{1/2}} \sim x^{2/3} \quad \text{for } x \ll 1$$

$$\sim \rho^{2/3}/3$$

White Dwarf : Solution

$\therefore \quad \frac{dm}{dr} = r^2 \rho \quad , \frac{d\rho}{dr} = -3m\rho^{1/3}/r^2$

we see, for v. small ρ, that $dm/dr \sim 0 \Rightarrow m \sim$ const.

$\therefore$ m becomes const. near surface (& beyond!)

clearly, $d\rho/dr < 0$ always .... so ρ must decrease.

$\therefore$ $\frac{d\rho}{dr} = -3m\rho^{1/3}/r^2$ can be integrated near surface

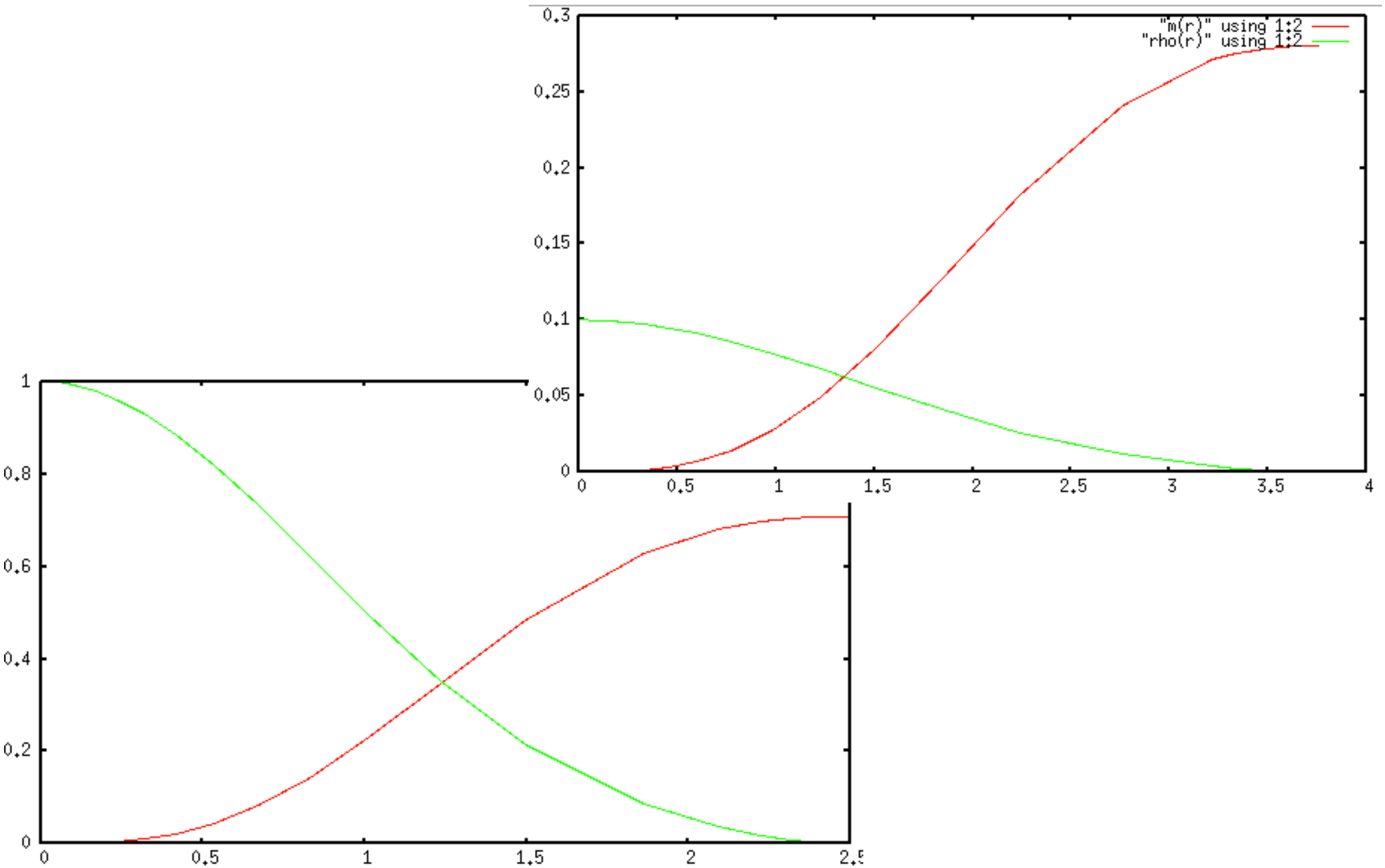
$$+3\rho^{-1/3} d\rho = -\frac{M}{r^2} dr \qquad \therefore \frac{\rho^{2/3}}{2} = \frac{M}{2r}$$

$$\therefore \rho \sim M^{3/2} \, r^{-3/2} \quad \text{near surface}$$

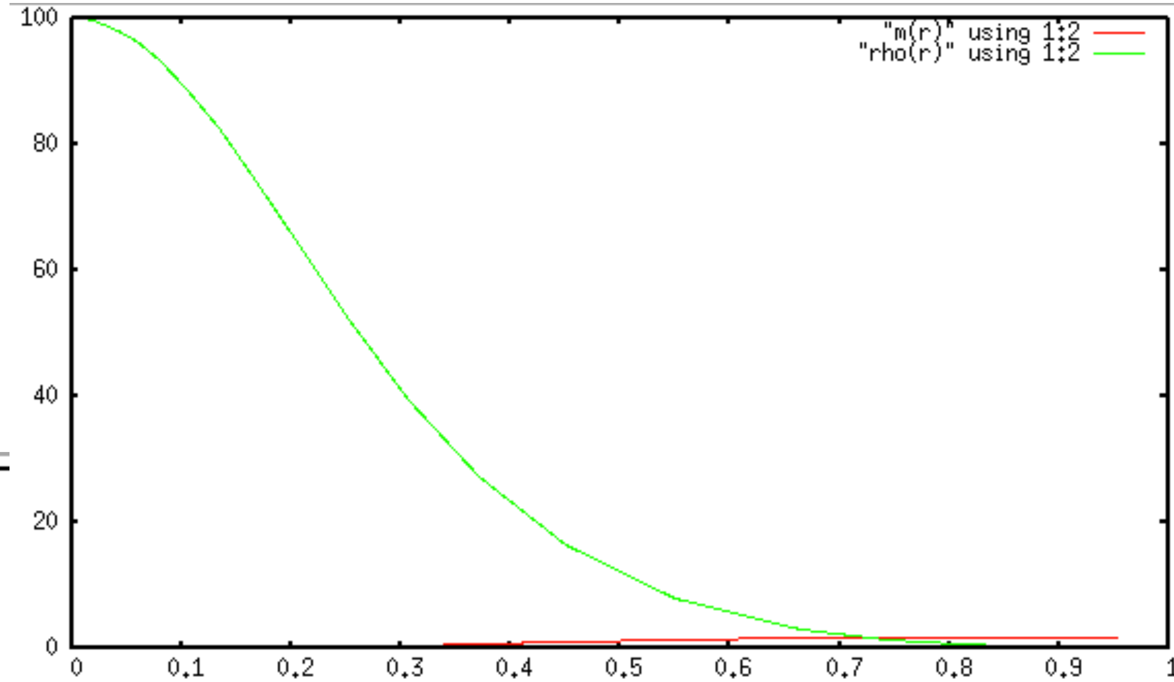
$$r^{-3/2} < \frac{1}{r}.$$

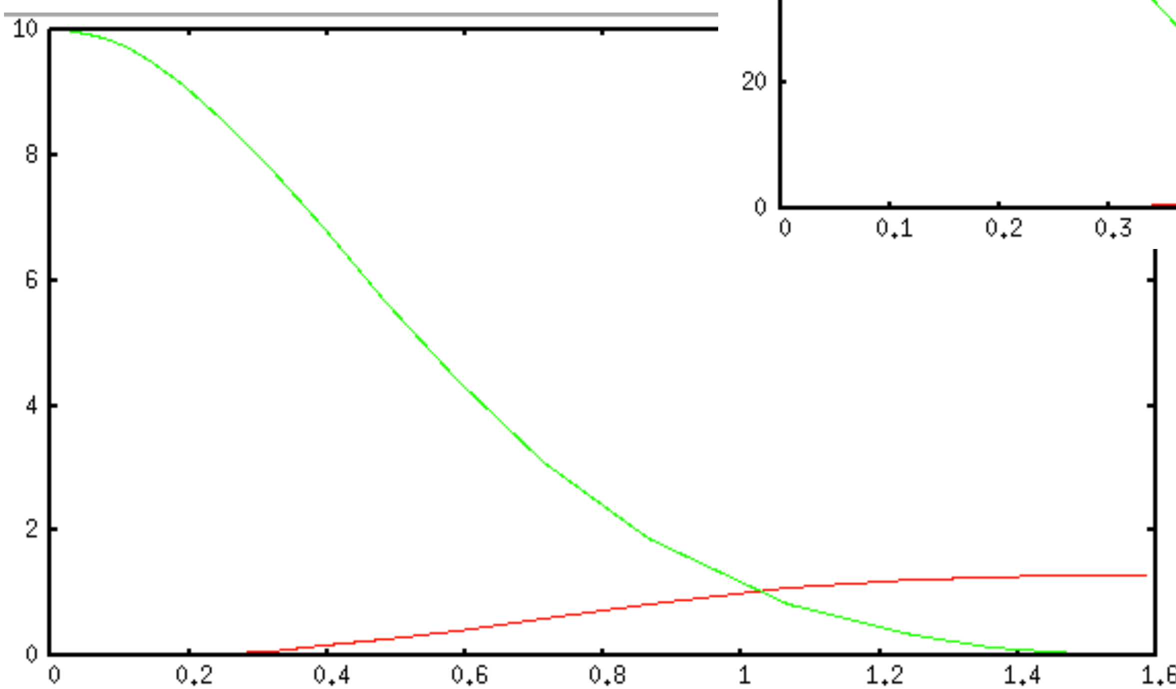
→ Surface well defined

White Dwarf : Solution



White Dwarf : Solution





White Dwarf : Solution

c) $\rho' \sim \dfrac{\bar{\rho}}{Y_e}$, $r' = \bar{r}\,Y_e$, $m' = \bar{M}\,Y_e^2$

.... scaling from calc which had $Y_e = 1$

now, for ^{12}C , $Y_e = \frac{1}{2}$

^{56}Fe $Y_e = 0.464$

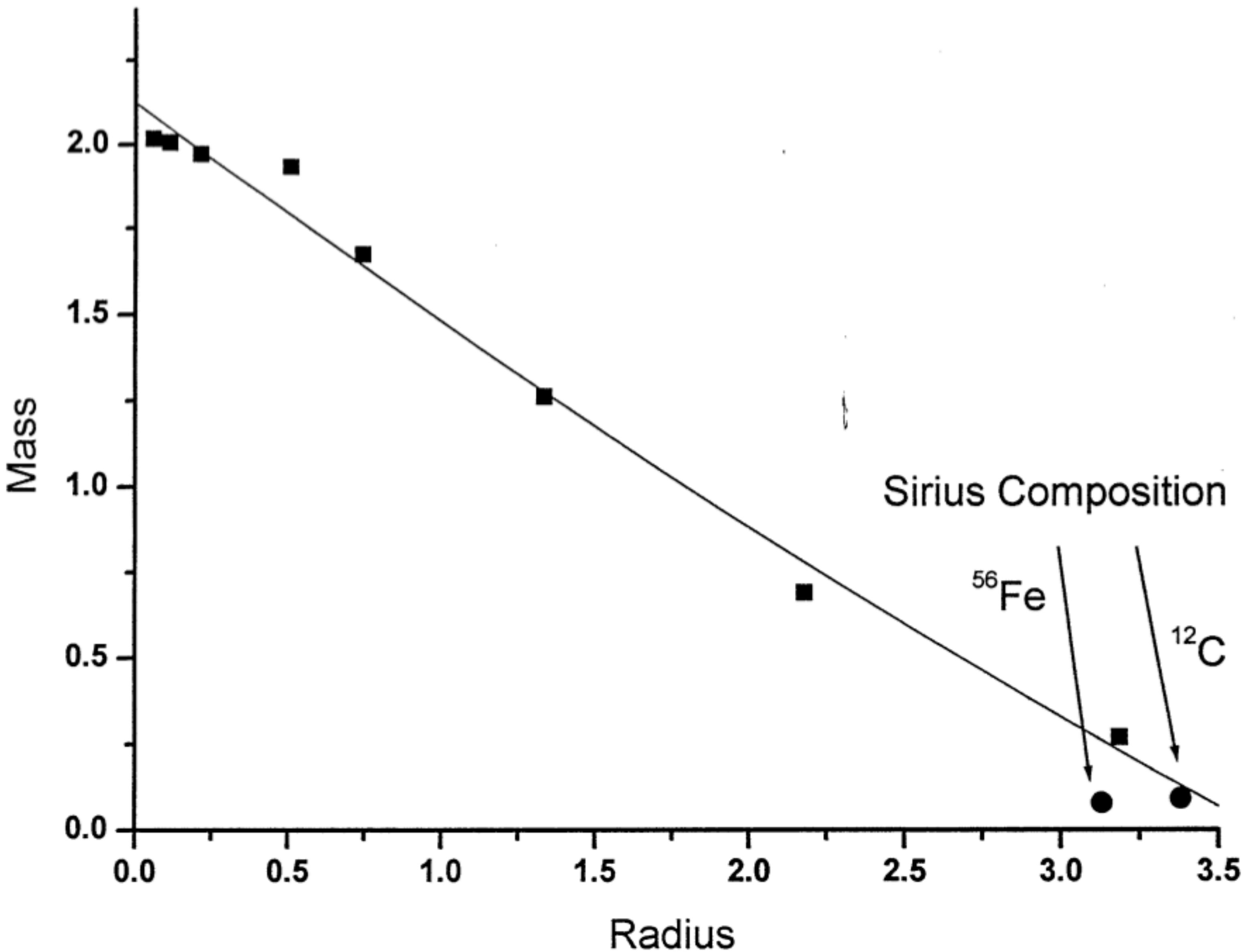
White Dwarf : Solution

$$\frac{R_0}{R_\odot} = 1.11 \times 10^{-2} \times Y_e$$

$$\frac{M_0}{M_\odot} = 2.86 \times Y_e^2$$

	$M (M_\odot)$	$R (R_\odot)$	
∴ Sirius B →	0.367	6.76	ie, same units as calculation ($Y_e = 1$)
Sirius B (^{12}C)	0.092	3.38	convert to $Y_e = 1$
Sirius B (^{56}Fe)	0.079	3.13	"

White Dwarf : Solution



Chandrasekhar limit

From Wikipedia, the free encyclopedia

The **Chandrasekhar limit** /tʃʌndrəˈʃeɪkɑr/ is the maximum mass of a stable white dwarf star. The limit was first published by Wilhelm Anderson and E. C. Stoner, and was named after Subrahmanyan Chandrasekhar, the Indian-American astrophysicist who improved upon the accuracy of the calculation in 1930, at the age of 19. This limit was initially ignored by the community of scientists because such a limit would logically require the existence of black holes, which were considered a scientific impossibility at the time. White dwarfs, unlike main sequence stars, resist gravitational collapse primarily through electron degeneracy pressure, rather than thermal pressure. The Chandrasekhar limit is the mass above which electron degeneracy pressure in the star's core is insufficient to balance the star's own gravitational self-attraction. Consequently, white dwarfs with masses greater than the limit undergo further gravitational collapse, evolving into a different type of stellar remnant, such as a neutron star or black hole. Those with masses under the limit remain stable as white dwarfs.[1]

The currently accepted value of the limit is about 1.44 $M_\odot$ (2.864 × 10^{30} kg).[2][3][4]

This model finds …… 2 M_0 * Y_e ~ 5 x 10^{30}kg

White Dwarf : Solution

e) $\rho = \frac{M}{V}$ (const. also, relativistic limit $x \gg 1$)

$$\frac{E}{V} = n_0 m_e x^3 \epsilon(x)$$

$$\epsilon(x) \sim \frac{3}{8x^3}\left[2x^4\right] \quad \text{when } x \gg 1$$

$$= \frac{3}{4} x$$

$$\therefore E/V = \frac{3 n_0 m_e}{4} \cdot \bar{\rho}^{4/3} \quad , \quad \rho = \rho_0 \bar{\rho}$$

White Dwarf : Solution

$$\therefore\ U = \int_0^R \left(\frac{E}{V}\right) 4\pi r^2 dr = \pi n_0 m_e \bar{\rho}^{4/3} R^3$$

$$= \pi n_0 m_e \left[\frac{M}{\frac{4}{3}\pi R^3 \rho_c}\right]^{4/3} R^3$$

$$= \pi n_0 m_e \left[\frac{3M}{4\pi\rho_c}\right]^{4/3} \cdot \frac{1}{R}$$

$$d\bar{m}/d\bar{r} = r^2 \bar{\rho}$$

$$m(r) = 4\pi \int_0^r \rho\, r'^2 dr'$$

$$= \frac{4\pi}{3} \rho r^3$$

White Dwarf : Solution

$$\therefore\ W = -\int_0^R \frac{G\,m(r)}{r}\,\rho\,4\pi r^2\,dr$$

$$= -G\,\frac{4\pi}{3}\rho^2 \cdot 4\pi \int r^4\,dr$$

$$= -G \cdot 3\left(\frac{4\pi}{3}\right)^2 \cdot \frac{M^2}{\left(\frac{4}{3}\pi R^3\right)^2} \cdot \frac{R^5}{5}$$

$$= -\frac{3}{5} G M^2 \frac{1}{R}$$

White Dwarf : Solution

$$(U + W) = \frac{1}{R}\left(a M^{4/3} - b M^{2}\right)$$, a , b some const.

$\therefore$ as $M \rightarrow M_{ch}$ ... $|W| > |U|$ $\Rightarrow$ collapse.

Computational Physics

Want you learned

- Computers are becoming MUCH more powerful
- Computational Physics is becoming MUCH more necessary
- You will be left behind if you don't develop the skills
- You will be very marketable if you do

Homework

- Do the C++ Tutorial and the White Dwarf Example
- Modify it yourself for Neutron Stars
- Continue to develop you self-taught computing skills.