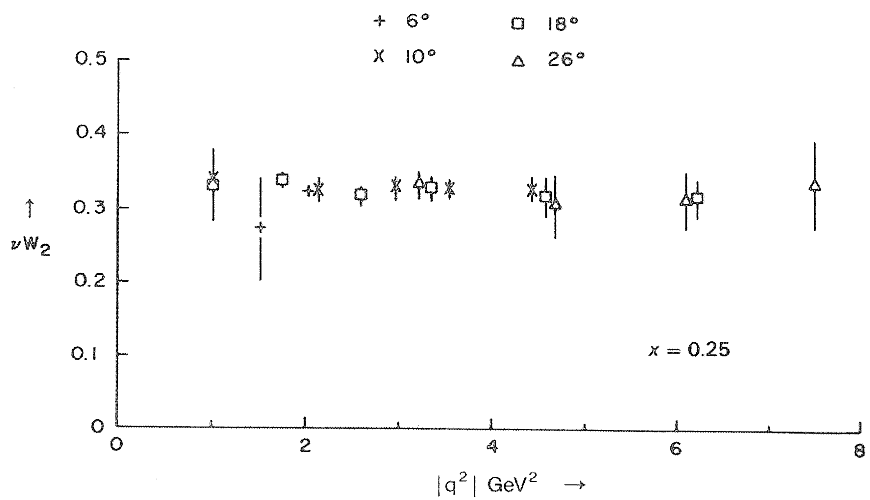


FIGURE 15.7
One of the (effective) structure functions (νW_2) in electron-proton scattering, which is expected to show scaling, as a function of $|q^2|$, for x held fixed at the value 0.25. (SLAC data)

18

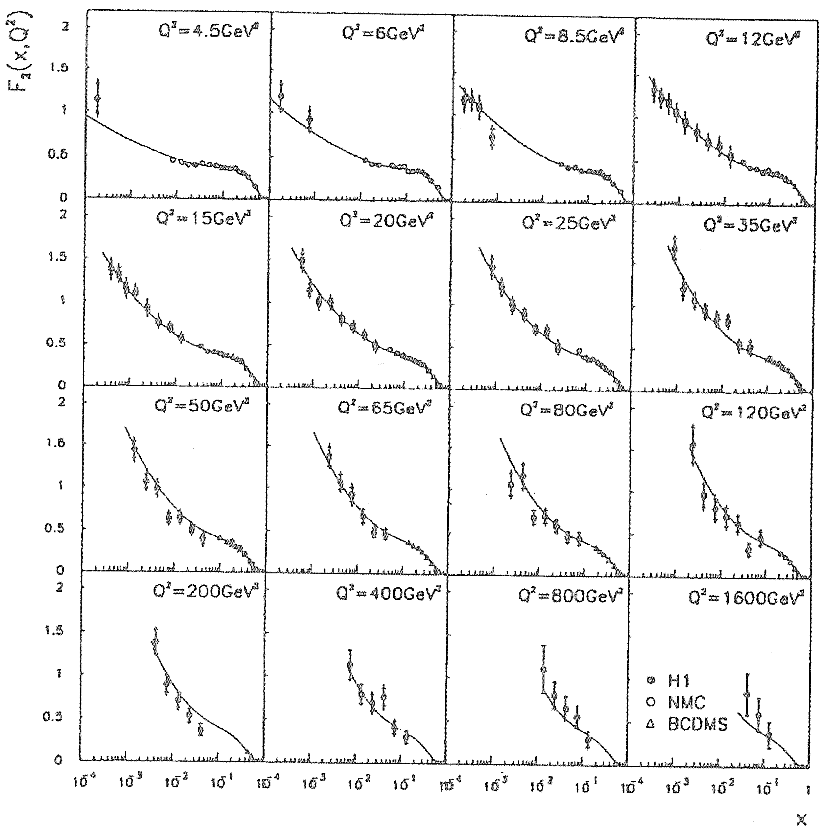


Fig. 10.7. The structure function $F_2(x, q^2)$ from Dainton, J. in *Proc. Workshop on Deep Inelastic Scattering and QCD* (eds. Laporte, JF. and Sirois, Y.). Editions de l'Ecole Polytechnique, Paris 1995

82

LECTURE 6: QCD II

SLAC experiments discovered point-like objects (constituents) inside the nucleon. Same story for all hadrons.

QUESTIONS:

1) Where does the $N_c = 3$ colour factor come from
2) Why does DIS suggest a "weak" interaction at short distances (large momentum)?
3) Why don't we see free quarks?
4) If the strong interaction is a gauge theory, Which are the gauge bosons?
5) If 4) which is the gauge group?
6) Does the theory render finite results as QED?

— x —

The QED Lagrangian, Eq. (III.39)

$$\mathcal{L}_{QED} = \bar{\Psi}(x)(i\gamma^\mu \partial_\mu - m_0)\Psi(x) - \frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) + J_\mu(x)A^\mu(x) \qquad (VI.1)$$

can be rewritten as

$$\mathcal{L}_{QED} = \bar{\Psi}(x)(i\gamma^\mu D_\mu(x) - m_0)\Psi(x) - \frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) \qquad (IV.2)$$

with

$$D_\mu(x) \equiv \partial_\mu + i q A_\mu(x) \quad \text{see (IV.3)}$$

the "covariant derivative".

$\mathcal{L}_{QED}$ "unzipped" explains: Atomic, Molecular, Condensed Matter Physics, Lasers, Chemistry, EM waves, Electric & Electronic Engineering, part of Biochemistry

Ingredients for "cooking" QCD:

1) strong interaction due to interaction among quarks, but flavour blind
2) Quarks come in three colours & six flavours. An experimental fact.
3) Gluons (massless) mediate the strong interaction among quarks & carry "colour" quantum numbers.
4) The unbroken local gauge group is an $SU(3)_c$, (c for colour) NOT TO BE CONFUSED WITH $SU(3)_f$ (f- for flavour) which is a manifestation of a broken global symmetry of the hadrons (bound states of quarks).
5) At short distances the strong interaction is "weak". Hence the quark-gluon coupling has the opposite behaviour as in QED

QED:

vacuum polarization

the closer to ~~the~~ the e⁻, the stronger the charge of the e⁻

6) At long distances the interaction becomes strong & there is no quark-gluon deconfinement (at $T=0$)

STEP 1.

Free matter (quark) Lagrangian: $\mathcal{L}^{(0)}_{QCD}(x)$

$$\mathcal{L}^{(0)}_{QCD}(x) = \sum_{i=u,d,s\ldots n_f} \sum_{a=1}^{3} \bar{\psi}^a_i(x)\,(i\gamma^\mu \partial_\mu - m^0_i)\,\psi^a_i(x) \quad (\text{VI}.2)$$

Local gauge transformation under (non-Abelian) $SU(3)_c$

$$\psi^a(x) \rightarrow \psi'^a(x) = U^{ab}(x)\,\psi^b(x), \quad (\text{VI}.3)$$

where flavour indexes have been omitted, and $a,b = 1,2,3$, and

$$U^{ab} = \left[e^{-ig\,T_j\,\omega_j(x)}\right]^{ab} \quad (\text{VI}.4)$$

$$j = 1,2\ldots N_c^2-1 = 8$$

For an infinitesimal gauge transformation

$$U^{ab} \simeq \mathbb{1}^{ab} - ig\,\left[T_j\,\omega_j(x)\right]^{ab} \quad (\text{VI}.5)$$

The objects T_j can be represented by 8 3×3 matrices with the following properties:

$$T_j^\dagger = T_j\,, \quad Tr(T_j)=0\,, \quad [T_i, T_j] = i f_{ijk} T_k \quad (\text{VI}.6)$$

and the normalization condition $Tr(T_i T_j) = \frac{1}{2}\delta_{ij}$.
The T_i are proportional to the Gell-Mann λ-matrices (the 3×3 extension of the Pauli matrices)

$$T_i = \frac{1}{2}\lambda_i \quad (\text{VI}.7)$$

The explicit expressions of the λ_i are:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}; \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \qquad (\overline{\text{VI}}.8)$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The structure constants f_{ijk} are given by

$$f_{ijk} = -\frac{i}{4} \operatorname{Tr}\{[\lambda_i, \lambda_j]\lambda_k\} \qquad (\overline{\text{VI}}.9)$$

or explicitly:

ijk	f_{ijk}	ijk	f_{ijk}	ijk	f_{ijk}
123	1	246	1/2	367	-1/2
147	1/2	257	1/2	458	$\sqrt{3}/2$
156	-1/2	345	1/2	678	$\sqrt{3}/2$

$(\overline{\text{VI}}.10)$

Omitting the colour indexes $(a, b = 1,2,3)$ the transformation of $\partial_\mu \psi(x)$ is

$$\partial_\mu \psi(x) \to \partial_\mu \psi'(x) = U(x)\partial_\mu \psi(x) + [\partial_\mu U(x)]\psi(x) \qquad (\overline{\text{VI}}.11)$$

Hence $\mathcal{L}^{(0)}_{QCD}$ becomes

$$\mathcal{L}^{(0)\prime}_{QCD} = \mathcal{L}^{(0)}_{QCD} + \bar{\psi}(x)\, i\gamma^\mu (U^\dagger(x)\partial_\mu U(x))\psi(x) \qquad (\overline{\text{VI}}.12)$$

which is obviously different from $\mathcal{L}^{(0)}_{QCD}$.
From our experience with QED, we add an interaction Lagrangian to $\mathcal{L}^{(0)}_{QCD}$

$$\mathcal{L}^{(1)}_{QCD} = \mathcal{L}^{(0)}_{QCD} + \mathcal{L}_I \equiv \mathcal{L}^{(0)}_{QCD} - g\,\bar{\psi}(x)\gamma^\mu \vec{A}_\mu(x)\psi(x), \qquad (\overline{\text{VI}}.13)$$

where

$$\vec{A}^\mu(x) \equiv A^\mu_j(x)\, T_j \quad (j = 1,2,\dots 8) \qquad (\overline{\text{VI}}.14)$$

and the transformation properties of $\vec{A}_\mu(x)$ will be fixed so that $\mathcal{L}_{QCD}$ is invariant under (VI.3). From (VI.14) it follows

$$\mathcal{L}'^{(1)}_{QCD} = \mathcal{L}^{(0)}_{QCD} + \bar\psi(x)\, i\gamma^\mu [U^\dagger(x)\,\partial_\mu U(x)]\,\psi(x) - g\,\bar\psi(x)\,U^\dagger(x)\,\gamma^\mu \vec{A}'_\mu(x)\,U(x)\,\psi(x) \qquad (\text{VI}.15)$$

Gauge invariance under $SU(3)_c$ then requires

$$\bar\psi(x)\, i\gamma^\mu [U^\dagger(x)\,\partial_\mu U(x)]\,\psi(x) - g\,\bar\psi(x)\,U^\dagger(x)\,\gamma^\mu \vec{A}'_\mu U(x)\,\psi(x) = -g\,\bar\psi(x)\,\gamma^\mu \vec{A}_\mu(x)\,\psi(x) \qquad (\text{VI}.16)$$

which uniquely fixes the transformation of $\vec{A}_\mu(x)$:

$$\Longrightarrow \quad \vec{A}_\mu(x) \to \vec{A}'_\mu(x) = \frac{i}{g}[\partial_\mu U(x)]\,U^\dagger(x) + U(x)\,A_\mu(x)\,U^\dagger(x) \qquad (\text{VI}.17)$$

—— x ——

<u>TUTORIAL</u>:

① Verify explicitly that (VI.17) is the correct transformation

② Assuming an infinitesimal gauge transformation i.e. Eq. (VI.5) show that

with
$$\psi'(x) = \psi(x) + \delta\psi(x),$$

$$\delta\psi^a(x) = -ig\,\omega_j(x)\,(T_j)^{ab}\,\psi^b(x)$$

$$\vec{A}'_\mu(x) \Rightarrow A_\mu(x) + \delta\vec{A}_\mu(x)$$

where
$$\delta\vec{A}_\mu(x) \equiv \delta A_{\mu i}^{(x)}\,T_i = \partial_\mu \omega_i(x)\,T_i - g f_{ijk}\,A_{\mu j}(x)\,\omega_k(x)\,T_i \qquad (\text{VI}.18)$$

$(i = 1, 2 \ldots 8)$

③ Show that
$$\delta\vec{A}_\mu(x) = \partial_\mu \omega^i(x)\,T^i + ig\,A_\mu^j(x)\,\omega^k(x)\,[T_j, T_k]$$

where $[T_j, T_k] = i f_{jki}\,T_i = i f_{ijk}\,T_i$ ..

Covariant derivative:

definition in QCD: $\vec{D}_\mu \equiv \partial_\mu + i g \vec{A}_\mu$ (VI.19)

where $\vec{D}_\mu \equiv D_{\mu i} T_i$, $\vec{A}_\mu = A_{\mu_i} T_i$

Using (VI.19) $\mathcal{L}^{(1)}_{QCD}$ can be written as

$$\mathcal{L}^{(1)}_{QCD} = \bar{\psi}(x) \left(i \gamma^\mu \vec{D}_\mu(x) - m_0 \right) \psi(x) \quad \text{(VI.20)}$$

TUTORIAL

(1) Show how to obtain (VI.20) (i.e. derive it using (VI.19)).

(2) Show that in QED

$$[D_\mu(x), D_\nu(x)] = i|e| F_{\mu\nu}(x) \quad \text{(VI.21)}$$

hint, apply the commutator to the Dirac field

$$[D_\mu(x), D_\nu(x)] \psi_D(x) = i|e| F_{\mu\nu}(x) \psi_D(x).$$

x

In analogy with (VI.21) we define in QCD

$$[\vec{D}_\mu(x), \vec{D}_\nu(x)] \psi(x) = i g \vec{F}_{\mu\nu}(x) \psi(x), \quad \text{(VI.22)}$$

where $\vec{D}_\mu(x) = D^i_\mu(x) T^i$, $\vec{F}_{\mu\nu}(x) = F^i_{\mu\nu}(x) T^i$

TUTORIAL:

Show that in QCD the gauge tensor $F_{\mu\nu}(x)$, while covariant, it is not invariant, i.e.

$$\vec{F}'_{\mu\nu}(x) = \vec{F}_{\mu\nu}(x) + i g \left(\vec{F}_{\mu\nu}(x) \vec{\omega} - \vec{\omega} \vec{F}_{\mu\nu}(x) \right) \quad \text{(VI.23)}$$

where $\vec{F}_{\mu\nu} = F_{\mu\nu} T_i$, and

$$\text{(VI.24)} \quad F^i_{\mu\nu}(x) = \partial_\mu A^i_\nu(x) - \partial_\nu A^i_\mu(x) - g f_{ijk} A^j_\mu(x) A^k_\nu(x)$$

In summary,

$$\mathcal{L}_{QCD} = \sum_{A=1}^{n_F} \bar{\psi}_a^{A}(x)\left(i\gamma^\mu \vec{D}_\mu(x) - m_A^0\right)\psi_a^A(x) - $$

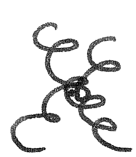
$$- \frac{1}{4}\sum_i F_{\mu\nu}^i(x)\, F_i^{\mu\nu}(x) \qquad (\mathrm{VI}.24)$$

where: $F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i - g f^{ijk} A_\mu^j A_\nu^k$

$$\vec{D}_\mu \psi_a^A(x) = \partial_\mu \psi_a^A(x) + \frac{i}{2} A_\mu^i(x)(\lambda_i)_{ab}\psi_b^A(x) \qquad (\mathrm{VI}.25)$$

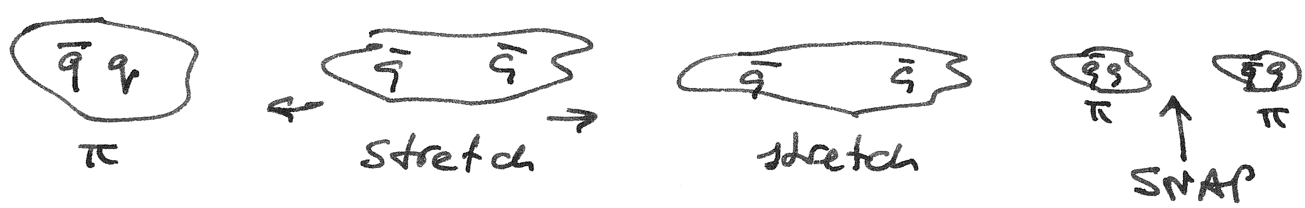
and we have changed the label of flavour into "$A = 1, 2, \ldots n_F$".

TUTORIAL:

Show that the gluon piece of $\mathcal{L}_{QCD}$ involves a) the gluon propagator , b) a three-gluon vertex: , and a four gluon vertex: . Hint: substitute the explicit expression for $F_{\mu\nu}^i(x)$ in (VI.25) into (VI.24).

CONFINEMENT:

Hadrons are colour singlet objects (no net colour) & quarks are permanently confined, e.g.



JET PRODUCTION in e^+e^- collisions

2 jet & 3 jet events.

90

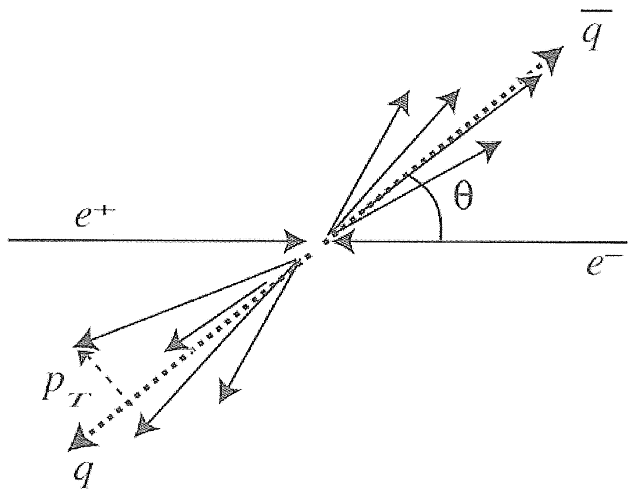


Fig. 6.1. Hadronisation of two quarks into jets.

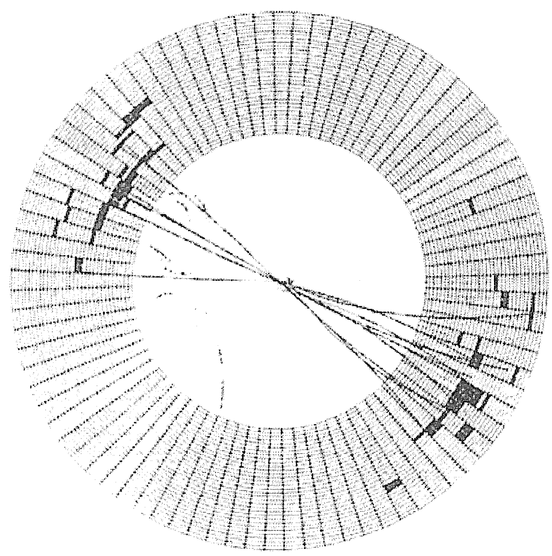


Fig. 6.2. Two-jet event in the JADE detector of the PETRA collider at DESY. (Naroska 1987)

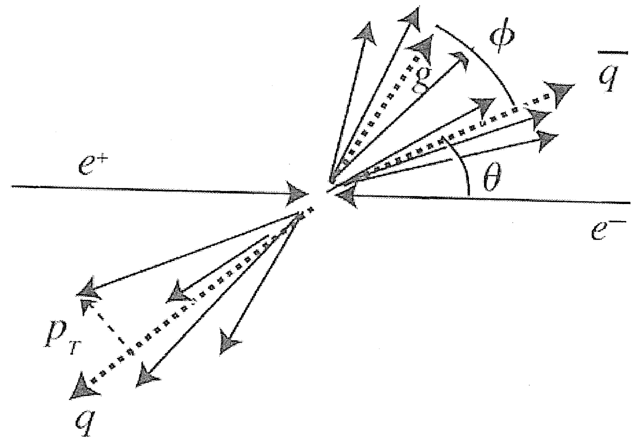


Fig. 6.5. Sketch of the gluon radiation.

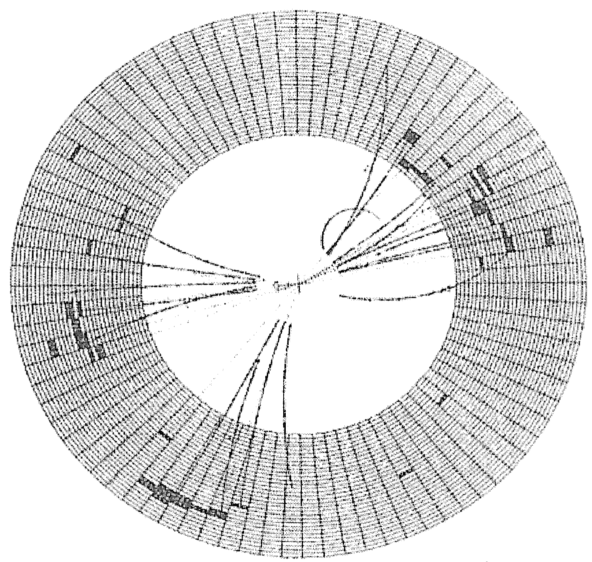


Fig. 6.6. Three-jet event at the JADE detector at the PETRA collider at DESY. (Naroska 1987)

LECTURE VII : RENORMALIZATION IN QFT

Green's Function:

The Green's function of a given differential operator is defined as

$$\mathcal{D}_x G(x-y) = (\pm \text{const})\,\delta^{(4)}(x-y), \quad (\text{VII}.1)$$

where $\mathcal{D}_x$ is a differential operator, x & y are space-time coordinates, and ($\pm$ const) is conventional. The delta function is defined as

$$\delta^{(4)}(x-y) = \frac{1}{(2\pi)^4}\int d^4k\; e^{-ik\cdot(x-y)} \quad (\text{VII}.2)$$

If the differential operator satisfies a differential equation, e.g.

$$\mathcal{D}\,\psi(x) = J(x), \quad (\text{VII}.3)$$

then the solution, $\psi(x)$, is given by

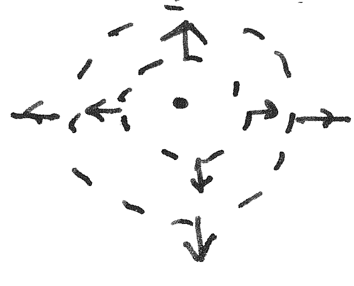
$$\psi(x) = \int d^4y\; G(x-y)\,J(y) \quad (\text{VII}.4)$$

up to ($\pm$ const). In fact, acting with $\mathcal{D}$ on ψ

$$\mathcal{D}_x\psi = \int d^4y\; \underbrace{\mathcal{D}_x G(x-y)}_{\delta^{(4)}(x-y)}\; J(y) = J(x), \text{ q.e.d.} \quad (\text{VII}.5)$$

Hence the Green's function "propagates" the disturbance produced by the source $J(y)$, according to Huyghens principle

A source of a spherical wave.



Each point on the spherical wave-front is a source of a spherical wave propagating from $J(y)$ outwards.

Often the explicit expression of $G(x-y)$ is complicated, and/or involves "special functions". It is better to work with the Fourier transform $G(k)$, i.e.

$$G(x-y) = \frac{1}{(2\pi)^4}\int d^4k \; e^{-ik\cdot(x-y)}\, G(k). \qquad (\underline{\text{VII}}.6)$$

<u>Example</u>: Klein-Gordon field $\varphi(x)$

$$(\partial^\mu \partial_\mu + m_0^2)\,\varphi(x) = J(x) \qquad (\underline{\text{VII}}.7)$$

From $(\underline{\text{VII}}.4)$

$$\varphi(x) = \int d^4y \; G(x-y)\, J(y)$$

applying $(\partial^\mu\partial_\mu + m_0^2)$ on both sides of the eqn.

$$(\partial^\mu\partial_\mu + m_0^2)\,\varphi(x) = \int d^4y\,(\partial^\mu\partial_\mu + m_0^2)\,G(x-y)\,J(y)$$

$$= \int d^4y\; \delta^{(4)}(x-y)\,J(y) = J(x) \qquad (\underline{\text{VII}}.8)$$

The Fourier transform is

$$(\partial^\mu\partial_\mu + m_0^2)\,G(x-y) = \delta^{(4)}(x-y) =$$

$$= \frac{1}{(2\pi)^4}\int d^4k\;(\partial^\mu\partial_\mu + m_0^2)\,e^{-ik\cdot(x-y)}\,G(k)$$

$$= \frac{1}{(2\pi)^4}\int d^4k\;\left[(-ik)^2 + m_0^2\right] e^{-ik\cdot(x-y)}\,G(k)$$

$$= \frac{1}{(2\pi)^4}\int d^4x\; e^{-ik\cdot(x-y)} \qquad (\underline{\text{VII}}.9)$$

Hence

$$\boxed{G(k) = \frac{-1}{k^2 - m_0^2}} \qquad (\underline{\text{VII}}.10)$$

If $m_0 = 0$, then Eq. (VII.7) is the D'Alembert (wave) equation for each component of the electromagnetic field $A(x)$. For quantum fields the definition of the propagator is:

$$\underset{y}{\otimes} \longrightarrow \underset{x}{\otimes} \; : \; \Delta_{KG}(x-y)$$

$$i\,\Delta_{KG}(x-y) \triangleq \langle 0 | T(\hat{\varphi}(x)\,\hat{\varphi}^{\dagger}(y)) | 0 \rangle \quad \text{(VII.11)}$$

where

$$T(\hat{A}(x)\,\hat{B}(y)) = \theta(x_0 - y_0)\,\hat{A}(x)\,\hat{B}(y) + \theta(y_0 - x_0)\,\hat{B}(y)\,\hat{A}(x) \quad \text{(VII.12)}$$

For the Klein-Gordon quantum field

$$\boxed{i\,\Delta_{KG}(k) = \frac{-\,i}{k^2 - m_0^2}} \quad \text{(VII.12)}$$

TUTORIAL

(1) Consider the propagator of a spin 1/2 particle $i\,S(x-y) = \langle 0 | T(\hat{\psi}(x)\,\hat{\bar{\psi}}(y)) | 0 \rangle$, such that $(i\gamma^{\mu}\partial_{\mu} - m_0)\,S(x-y) = \delta^{(4)}(x-y)$. Show that in p-space

$$\boxed{i\,S(p) = \frac{i}{\not{p} - m_0} = i\,\frac{(\not{p} + m_0)}{p^2 - m_0^2}} \quad \text{(VII.13)}$$

(2) Consider the propagator of a photon, i.e. $i\,D_{\mu\nu}(x-y) = \langle 0 | T(\hat{A}_{\mu}(x)\,\hat{A}_{\nu}(y)) | 0 \rangle$, such that

$$\Box^2 D_{\mu\nu}(x-y) = T_{\mu\nu}\,\delta^{(4)}(x-y), \text{ and with}$$

$$D_{\mu\nu}(x-y) = T_{\mu\nu}\,D(x-y)$$

find that

$$\boxed{i\,D(q^2) = \frac{-i}{q^2}} \quad \text{(VII.14)}$$

The tensor $T_{\mu\nu}$ above enters through the relation for the polarization vectors:

$$\sum_{\lambda=1}^{2} e_\mu(q,\lambda)\, e_\nu(q,\lambda) = -g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \qquad (\text{VII}.15)$$

(see page 53 and the last equation generalized from 3-dimensions to four.).

Using (VII.14) & (VII.15)

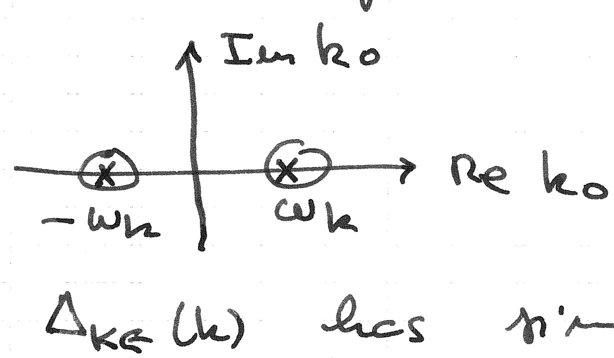
$$\boxed{i D_{\mu\nu}(q^2) = \frac{-i}{q^2}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)} \qquad (\text{VII}.16)$$

Notice that $q_\mu D_{\mu\nu} = q_\nu D_{\mu\nu} = q_\mu q_\nu D_{\mu\nu} = 0$.

<u>Interpret this!</u> Hint: recall properties of the photon (& of EM waves).—

—— x ——

Notice that all three propagators are singular at $k^2 = m_0^2$, $p^2 = m_0^2$, & $q^2 = 0$, i.e. on mass shell. To regularize & isolate the singularity one extends the propagators to the complex momentum space: e.g.



$$k^2 - m_0^2 = k_0^2 - \vec{k}^2 - m_0^2$$
$$= (k_0 + \omega_k)(k_0 - \omega_k)$$
with $\omega_k = +\sqrt{\vec{k}^2 + m_0^2}$

$\Delta_{KG}(k)$ has simple poles @ $k_0 = \pm\omega_k$

One needs to shift the poles by an amount ϵ, and follow the steps:

i) Cauchy's theorem in the complex k-plane for a function $f(k_0)$ which has poles:

$$\oint dk_0\, f(k_0) = 2\pi i \sum_i \text{Residues of } f(k_0) \qquad (\text{VII}.17)$$

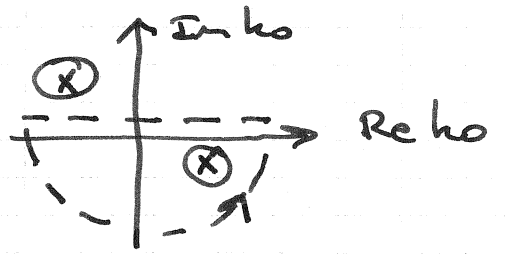
counter-clockwise integral (see Eq. VII.6)

$$\sum_i \text{Res. } f(k_0) = \lim_{k_0 \to \omega_k} (k_0 - \omega_k) \frac{e^{-ik\cdot(x-y)}}{(k_0+\omega_k)(k_0-\omega_k)}$$

$$+ \lim_{k_0 \to -\omega_k} (k_0 + \omega_k) \frac{e^{-ik\cdot(x-y)}}{(k_0+\omega_k)(k_0-\omega_k)} \qquad (\text{VII}.18)$$

Boundary conditions: If $x^0 > y^0$ only positive energy solutions propagate into the future, & if $x^0 < y^0$ only negative energy solutions propagate into the past. Hence,

$$\oint dk_0\, f(k_0) = \int_{-\infty}^{\infty} dk_0\, f(k_0) + \int_C dk_0\, f(k_0) \qquad (\text{VII}.19)$$

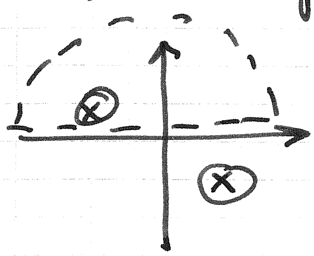


For $x^0 > y^0$

$$e^{-ik_0(x_0-y_0)} = e^{-i(k_{0R}+ik_{0I})(x^0-y^0)} = e^{-ik_{0R}(x_0-y_0)}\, e^{k_{0I}(x_0-y_0)}$$

Hence, for $x^0 > y^0$ choose the integration contour shown above, so that $e^{k_{0I}(x_0-y_0)} = e^{-|k_{0I}|(x_0-y_0)} \to 0$ asymptotically, and the pole has been shifted downwards so that only one residue contributes.

For $x^0 < y^0$ the contour is closed in the upper-half plane. All propagators are conventionally written as



$$i\Delta_{KG}(k) = \frac{-i}{k^2 - m_0^2 + i\epsilon} \qquad (\text{VII}.20)$$

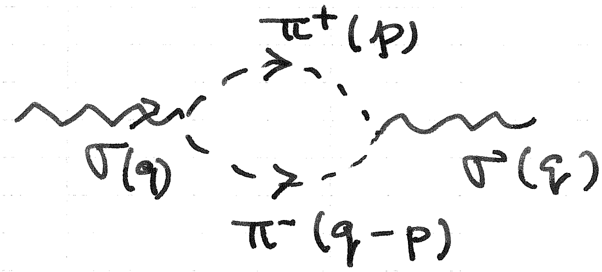
and similarly for $S(p)$ & $D(q)$

The addition of $i\epsilon$ in the denominator is a reminder of the procedure just explained.

— x —

INFINITIES AT NEXT-TO-LEADING (NL) ORDER

Consider the σ-model at the one-loop level:



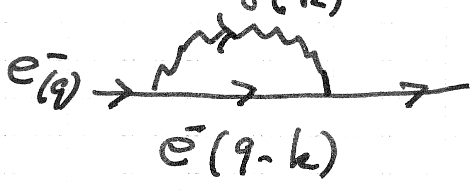
the S-matrix will involve the integral

$$I \propto \int d^4p \; \Delta_{KG}(p)\, \Delta_{KG}(q-p) \qquad (\text{VII}.21)$$

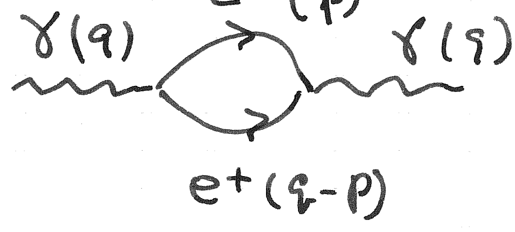
$$\propto \int d^4p \; \frac{1}{p^2 - m_0^2 + i\epsilon} \; \frac{1}{(q-p)^2 - m_0^2 + i\epsilon}$$

which is logarithmically divergent.

Other examples:

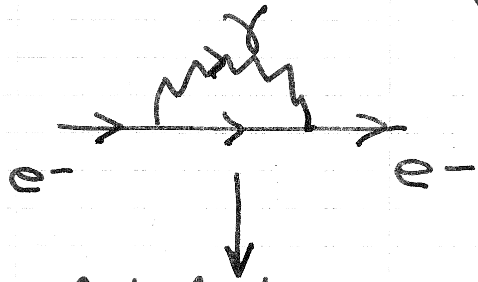


;

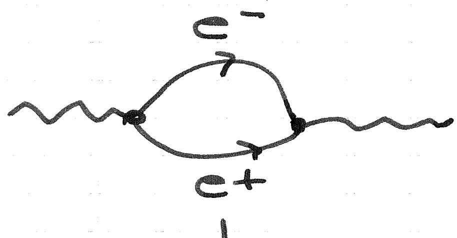


also logarithmically divergent.

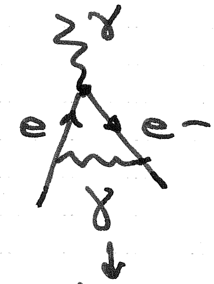
<u>Divergences</u>: A feature of QFT, & its ultraviolet (high momentum, short-distance) behaviour. A consequence of vacuum fluctuations. In QED the three basic divergent diagrams are (to one-loop level):



related to m_{e^-}



related to $|e|$ (charge)



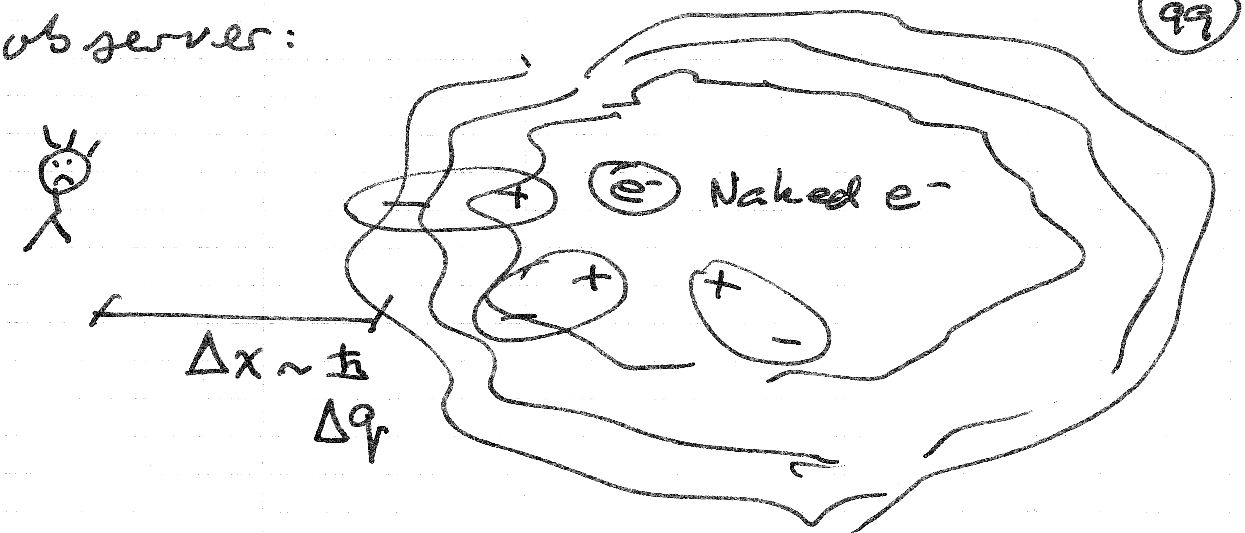
related to $|\vec{\mu}_e|$ (magnetic moment)

There are, obviously higher order vacuum fluctuations, e.g.

etc. -

Because of vacuum fluctuations, mass, charge (coupling) & eventually magnetic moment are affected by these fluctuations ∴ THERE IS NO SUCH A THING AS A "BARE" PARTICLE.

All particles in QFT are "dressed" by interactions arising from vacuum fluctuations. HENCE: m_{e^-}, $|e|$, $|\vec{\mu}_e|$ depend on the scale (distance) from the observer:



$m_{e^-} \Rightarrow m_{e^-}(q^2)$; $|e^-| \Rightarrow |e^-|(q^2)$, etc.

<u>WAY</u> <u>FORWARD</u>

1) <u>REGULARIZATION</u> : Isolate singularity (responsible for the divergences)

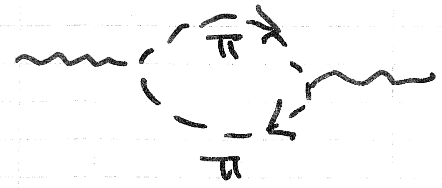
2) <u>RENORMALIZATION</u> : Render finite all divergent graphs by a redefinition of mass, charge (magnetic moment)

fixed a scale to "measure" $m, |e|, |\vec{\mu}_e|$ ...

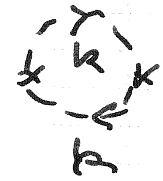
<u>Renormalizable theories</u>: Require only renormalization of the leading order diagrams. All other higher order diagrams will give finite answers. Examples: QED, QCD, Electroweak Theory.

<u>Non-renormalizable theories</u> : At each order in perturbation theory need to render finite the divergences. In principle need an ∞ number of rescalings.

Cutoff Regularization:



& consider only the loop



$$I = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{k^2 - m^2 + i\epsilon} \qquad (\text{VII}.22)$$

We omit the $i\epsilon$ from now on, and use Euclidean momentum for convenience

$$\begin{cases} k \to k^0 \to i k^0_E \\ k^2 = k^\mu k_\mu = -k_E^2 \end{cases} \qquad (\text{VII}.23)$$

$$I = -i \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{(k_E^2 + m^2)^2}, \quad \text{where} \qquad (\text{VII}.24)$$

$$d^4 k_E = k_E \, d^3 k_E = k_E^3 \, dk_E \, d\Omega \qquad (\text{VII}.25)$$

where $d\Omega = \sin^2\theta_1 \, d\theta_1 \, \sin\theta_2 \, d\theta_2 \, d\varphi$

$$\int d\Omega = \int_0^\pi d\theta_1 \sin^2\theta_1 \int_0^\pi d\theta_2 \sin\theta_2 \int_0^{2\pi} d\varphi \qquad (\text{VII}.26)$$

$$\int d\Omega = 2\pi^2 \qquad (\text{VII}.27)$$

$$I = \frac{-i}{(2\pi)^4} (2\pi^2) \int_0^\infty \frac{k_E^3}{(k_E^2 + m^2)^2} \; ; \quad y \equiv k_E^2$$

$$= \frac{-i}{(2\pi)^4} \frac{(2\pi^2)}{2} \int_0^\infty dy \, \frac{y + m^2 - m^2}{(y + m^2)^2} \; ; \quad \int_0^\infty \to \int_0^{\Lambda^2} \text{ with } \Lambda \to \infty$$

$$= \frac{-i}{(2\pi)^4} \frac{(2\pi^2)}{2} \lim_{\Lambda \to \infty} \left[\ln\left(1 + \frac{\Lambda^2}{m^2}\right) - 1 \right]$$

$$= \frac{-i}{(2\pi)^4} \frac{(2\pi^2)}{2} \lim_{\Lambda \to \infty} \left[\ln\left(\frac{\Lambda^2}{m^2}\right) + \mathcal{O}\left(\frac{1}{\Lambda^2}\right) - 1 \right] \qquad (\text{VII}.28)$$

The divergent part has been isolated, what remains is to renormalize (rescale). But we will not use method of regularization as it is not Lorentz covariant. Instead, we describe

<u>DIMENSIONAL REGULARIZATION</u>:

Integration in $D = 4 - \epsilon$ dimensions, with $\epsilon \to 0$ at the end.

<u>Preliminary</u>:

Consider $$\sum_{n=1}^{\infty} n \equiv S \qquad (\text{VII}.29)$$

Clearly $S = \infty$, but it has a finite part: $-\frac{1}{12}$

Euler-McLaurin summation formula:

$$\sum_{n=1}^{\infty}{}' F(n) = \int_0^{\infty} F(n)\, dn - \frac{1}{2} F(0) - \frac{1}{12} F'(0) + \frac{1}{720} F'''(0) + \dots \qquad (\text{VII}.20)$$

Using this formula

$$S \equiv \sum_{n=1}^{\infty} n = \underbrace{\int_0^{\infty} n\, dn}_{\text{divergent}} - \frac{1}{12} \qquad (\text{VII}.21)$$

Regularize by e.g. considering

$$\lim_{\epsilon \to 0} \sum_{n=1}^{\infty} n e^{-\epsilon n} = \int_0^{\infty} n e^{-\epsilon n}\, dn - \frac{1}{12} + \frac{1}{720} 3\epsilon^2 + \dots$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} - \frac{1}{12} + \mathcal{O}(\epsilon^2) \qquad (\text{VII}.22)$$

isolated singularity

we could also consider a different regulator, i.e. a different regularization scheme, e.g.

$$\lim_{\epsilon\to 0} \sum_{n=1}^{\infty} n\, e^{-\epsilon n^2} = \underset{\epsilon\to 0}{\mathcal{L}} \frac{1}{2\epsilon} - \frac{1}{12} \qquad (\text{VII}.23)$$

The finite part is the same !! In fact, it is INDEPENDENT of the regularization scheme !

Consider the formula:

$$I = \int x^n\,dx = \frac{x^{n+1}}{n+1} + \text{const.} \quad (\text{for } n \neq -1). \quad (\text{VII}.24)$$

Question: what happens if $n=-1$: regularize

$$I_{-1} \equiv \int x^{-1+\epsilon}\,dx = \frac{x^{(-1+\epsilon)+1}}{(-1+\epsilon)+1} + \underbrace{C}_{\text{Constant}} = \frac{1}{\epsilon}\underbrace{x^{\epsilon}}_{1+\epsilon\ln x+\mathcal{O}(\epsilon^2)} + \text{const.}$$

$$= \frac{1}{\epsilon} + \ln x + \mathcal{O}(\epsilon^2) \qquad (\text{VII}.25)$$

If a definite integral is computed, then

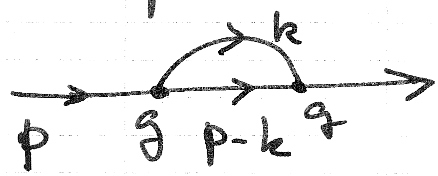
$$I_{-1}\Big|_a^b = \int_a^b x^{-1+\epsilon}\,dx = \underbrace{\left(\frac{1}{\epsilon} - \frac{1}{\epsilon}\right)}_{=0} + \ln(b/a) \quad (\text{VII}.26)$$

Choosing two scales (a,b) eliminates the singularity, and the result is finite.

This is the essence of (a) regularization & (b) renormalization, i.e. isolating the singularity & eliminating it through rescaling.

Feynman Parameterization:

Let us consider the following Feynman diagram for scalar particles



$$\begin{cases} k^2 = m_1^2 \\ p^2 = m_2^2 = (p-k)^2 \end{cases}$$

$$\Pi(p^2) \equiv g^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(p-k)^2 - m_2^2 + i\epsilon]} \frac{1}{(k^2 - m_1^2 + i\epsilon)} \qquad (\text{VII}.27)$$

Feynman's trick is to convert products of propagators into a single term, viz.

TUTORIAL

1) Show that $\dfrac{1}{ab} = \dfrac{1}{b-a} \displaystyle\int_a^b \frac{dt}{t^2}$ (VII.28) and

choose $t = b + (a-b)z$, $dt = (a-b)\,dz$, to show

2) $$\frac{1}{ab} = \int_0^1 \frac{dz}{[b + (a-b)z]^2}, \qquad (\text{VII}.29)$$

and, in general show

3) $$\frac{1}{A_1 A_2 \cdots A_n} = \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 dx_n \, \frac{\delta\left(\sum_i^n x_i - 1\right)(n-1)!}{[x_1 A_1 + x_2 A_2 + \cdots + x_n A_n]^n}$$ (VII.30)

Hint: take n- derivatives with respect to $\underline{a}$

$$\frac{d}{da}\left(\frac{1}{ab}\right) = -\frac{1}{aab} = -2\int_0^1 \frac{z\,dz}{[b + (a-b)z]^3}.$$

(VII.31)

Using (VII.30), $\mathrm{I}(p^2)$ in (VII.27) becomes

$$\mathrm{I}(p^2) = \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx_1 \int_0^1 dx_2 \frac{\delta(x_1 + x_2 - 1)}{\left\{x_1(k^2 - m_1^2) + x_2\left[(p-k)^2 - m_2^2\right]\right\}^2} \qquad \text{(VII.32)}$$

TUTORIAL

Show that

$$x_1(k^2 - m_1^2) + x_2\left[(p-k)^2 - m_2^2\right] = \qquad \text{(VII.33)}$$
$$= \ell^2 - \Delta(p^2)$$

where

$$\ell \equiv k - x_2 p \qquad \text{(VII.34)}$$

$$\Delta(p^2) = x_1 m_1^2 + x_2^2 p^2 - x_2(p^2 - m_2^2) \qquad \text{(VII.35)}$$

hint: add & subtract $x_2^2 p^2$ to complete a square $[(k - x_2 p)^2]$.

From the δ-function in (VII.32) show that

$$\Delta(p^2) = x_1 m_1^2 - x_1(1-x_1)p^2 + (1-x_1) m_2^2 \qquad \text{(VII.36)}$$

— x —

Hence

$$\mathrm{I}(p^2) = \int_0^1 dx_1 \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\left[\ell^2 - \Delta(p^2) + i\epsilon\right]^2} \qquad \text{(VII.37)}$$

where we restored the $i\epsilon$ in the denominator.

DIMENSIONAL REGULARIZATION

Instead of integrating in $D=4$ dimensions, one integrates in $D=4-\epsilon$ dimensions, and at the end of the calculation the singularity, always of the form $1/\epsilon$, will be isolated.

Master formula:

$$I_n \equiv \mu^{4-D} \int \frac{d^D k}{(k^2 - S + i\epsilon)^n} = i\, \pi^{D/2} (-)^n \frac{\Gamma(n - D/2)}{\Gamma(n)} \frac{1}{(S/\mu^2)^{n-\frac{D}{2}}} \qquad (\text{VII}.38)$$

$$(n > D/2)$$

where S is a function of dimension -2, and the parameter μ^2 is introduced for dimensional reasons (S/μ^2 is dimensionless)

Several properties of the Gamma function are needed:

Γ- FUNCTION

$$\Gamma(z+1) = z! = z\,\Gamma(z) = (z-1)!$$

Hence $\Gamma(z)$ is the analytic continuation of the factorial ($z!$ has meaning only for $z=$ integer; for non-integer values $\Gamma(z)$ is the extrapolating function)

$$\Gamma(z)\,\Gamma(1-z) = -z\,\Gamma(-z)\,\Gamma(z) = \pi \operatorname{cosec}(\pi z)$$

$$\lim_{\epsilon \to 0} \Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E \qquad (\gamma_E = 0.57721\ldots)$$

$$\Gamma(\epsilon) = \frac{1}{\epsilon}\,\Gamma(1+\epsilon) = \frac{1}{\epsilon}\left[1 + \epsilon\,\psi(1)\right],$$

where

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

is the digamma function, with $\psi(1) = -\gamma_E$.

$\Gamma(z)$ has simple poles at $z = 0, -1, -2, \ldots$ but is analytic everywhere else.

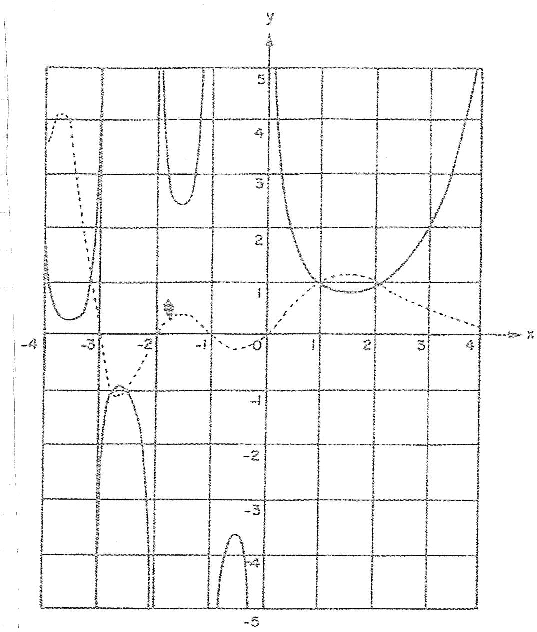


FIGURE 6.1. *Gamma function.* *

The master integral, (VII. 38), for $n=2$ is

$$I_2(s) = i\, \pi^{(2-\epsilon/2)} (-)^n \frac{\Gamma(n-2+\epsilon/2)}{\Gamma(n)} \frac{1}{(s/\mu^2)^{n-2+\frac{\epsilon}{2}}} \qquad \text{(VII. 39)}$$

it is <u>IMPORTANT</u> to take $\lim_{\epsilon \to 0}$ <u>only</u> at the end of the calculation, once all expansions have been made, e.g.

$$\Gamma(\epsilon/2) = \frac{2}{\epsilon} - \gamma_E \;;\; \left(\frac{s}{\mu^2}\right)^{-\epsilon/2} = 1 - \frac{\epsilon}{2} \ln\left(\frac{s}{\mu^2}\right) + \mathcal{O}(\epsilon^2)$$

$$\pi^{-\epsilon/2} = 1 - \frac{\epsilon}{2} \ln \pi + \mathcal{O}(\epsilon^2)$$

$$\Longrightarrow \quad I_2(s) = 2i\, \pi^2 \left\{ \frac{1}{\epsilon} - \frac{\gamma_E}{2} - \frac{1}{2} \ln\left(\frac{\pi s}{\mu^2}\right) \right\} \qquad \text{(VII. 40)}$$

Once the regularization is completed, one chooses a scale to define the divergent integral, e.g. s_0, and then it follows that

$$I_2(s) - I_2(s_0) = -i\pi^2 \ln(s/s_0) + \mathcal{O}(\epsilon^2) \qquad (\mathrm{VII}.41)$$

which is a finite quantity, i.e. a renormalized quantity. For instance, for the one-loop function (VII.27), using dimensional regularization in (VII.37) gives

$$\Pi(p^2) = \int_0^1 dx_1 \; I_0(p^2), \quad \text{with}$$

$$I_0(p^2) = \int \frac{d^4 l}{(2\pi)^4} \frac{1}{[l^2 - \Delta(p^2) + i\epsilon]^2}$$

$$= -\frac{4\pi^2}{(2\pi)^4}\left\{\frac{1}{6} + \frac{i}{4}\left(\gamma_E + \ln \frac{\pi\Delta(p^2)}{\mu^2}\right)\right\} \qquad (\mathrm{VII}.42)$$

Renormalizing @ 0 gives

$$I_0(p^2) - I_0(0) = -\frac{i}{(2\pi)^4}\pi^2 \ln\left[\frac{\Delta(p^2)}{\Delta(0)}\right] \qquad (\mathrm{VII}.43)$$

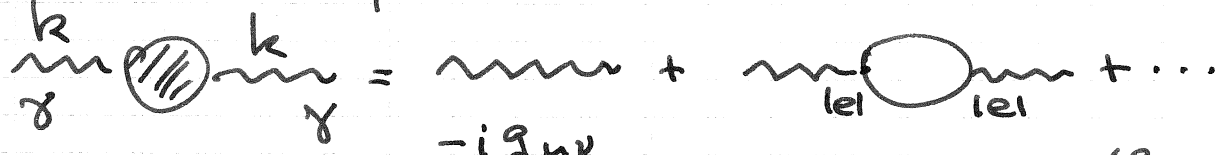
which is finite. Any other value of s could obviously be chosen as the scale to "measure" $I_0(p^2)$.–

Renormalization of QED

Only a brief, abridged, version will be discussed (see literature for details). The three "primitive" divergences in need of regularization & renormalization are the ones on page 98.

page 98

Vacuum Polarization: defines the coupling (electric charge $|e|$)



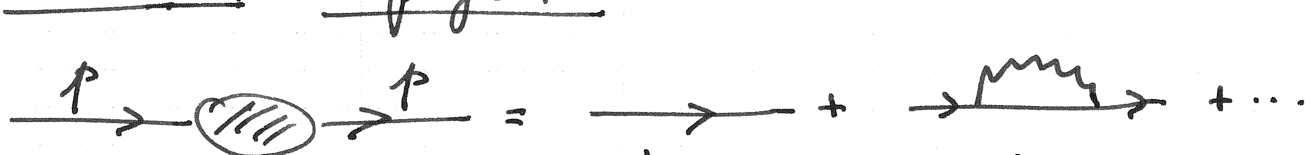
$$\frac{-i g_{\mu\nu}}{k^2+i\epsilon} + \frac{-i g_{\mu\alpha}}{k^2+i\epsilon}\, i e_0^2\, \Pi^{\alpha\beta}(k^2)\, \frac{(-i) g_{\beta\nu}}{k^2+i\epsilon} \cdots$$

where $\Pi^{\alpha\beta}(k^2)$ is log divergent. One usually chooses $k^2=0$ as the scale to "measure" the electric charge, but any other scale is equally valid. With

$$\alpha_{EM} = \frac{|e|^2}{4\pi\epsilon_0\hbar c} \;(\text{in SI units}) = \frac{1}{137.035\ldots} \quad @\, q^2=0$$

$$\alpha_{EM}(1\,GeV^2) \approx \frac{1}{136}, \quad \alpha_{EM}(M_W^2) \approx \frac{1}{128} \;\ldots$$

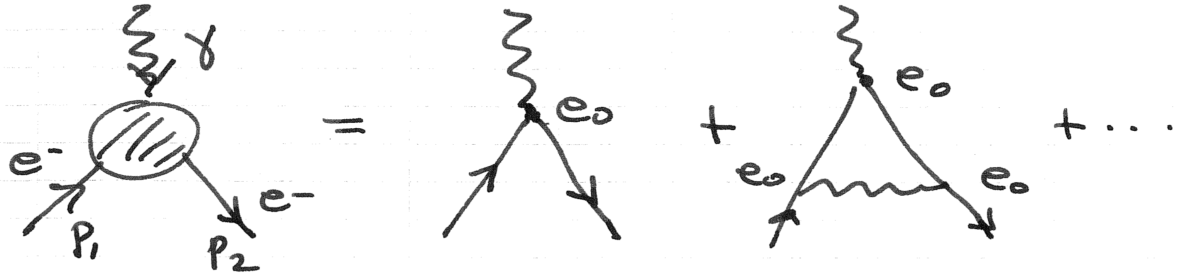
Electron Propagator



$$\frac{i}{\not p - m_0 + i\epsilon} + \frac{i}{\not p - m_0 + e_0^2 \Sigma(p) + i\epsilon} + \ldots$$

Where $\Sigma(p)$ is log divergent. Usually one performs on-mass-shell renormalization, i.e. the scale is m_e^2 itself. Since the e^- exists as a free particle this is ok.

VERTEX function



$$i e_0 \gamma_\mu \qquad\qquad i e_0^3 \Lambda_\mu (p_1, p_2)$$

where $\Lambda_\mu(p_1, p_2)$ is divergent. This term is needed to complete the renormalization programme, together with external line renormalizations. Once completed a NLO (next-to-leading order) it can be shown that results hold at all orders in perturbation theory, i.e. it is not necessary to renormalize again at NNLO (next-to-next-to-leading order), etc.- Only non-renormalizable theories require this at each order.

— x —

110

BIBLIOGRAPHY

1) Relativistic Quantum Mechanics, J. D. Bjorken and S. D. Drell, Mc Graw Hill.
2) Elementary Particles and Their Interactions, Quang Ho-Kim and Pham Xuan Yem, Springer Verlag.
3) Lectures on Quantum Field Theory, A. Das, World Scientific.
4) An Introduction to Quantum Field Theory, M. E. Peskin and D. V. Schroeder, Adisson Wesley.
5) The Theory of Fundamental Processes, R. P. Feynman, Benjamin/Cummins.
6) Quantum Electrodynamics, R. P. Feynman, Benjamin/Cummins.
7) Renormalization, L. M. Brown (editor), Spronger Verlag.
8) Quantum Field Theory (Revised Edition), F. Mandl and G. Shaw, John Wiley.
9) Relativistic Quantum Mechanics and Introduction to Field Theory, F. J. Yndurain, Springer Verlag.
10) Diagrammatica, M. Veltman, Cambridge University Press.
11) Quantum Field Theory, C. Itzykson and J.-B. Zuber, Dover.
12) Lectures on QED and QCD, A. Grozin, World Scientific.
13) The Quantum Theory of Fields, Vols. I, II, III, S. Weinberg, Cambridge University Press.
14) Modern Quantum Field Theory, T. Banks, Cambridge University Press.
15) Quantum Field Theory in a Nutshell, A. Zee, Princeton University Press.
16) Quantum Chromodynamics, Lecture Notes in Physics, Volume 118, J. L. Alonso and R. Tarrach (editors), Springer Verlag.
17) Thermal Field Theory, M. Le Bellac, Cambridge University Press.
18) Finite Temperature Field Theory, J. I. Kapusta, Cambridge University Press.
19) Finite Temperature Field Theory, A. Das, World Scientific.
20) Introduction to Quantum Field Theory, S. J. Chang, World Scientific.
21) Field Theory: A Modern Primer, P. Ramond, Benjamin/Cummins.
22) Field Theory: A Path Integral Approach, A. Das, World Scientific.

GRAMMATRIX

A COLLECTION OF FORMULAS FOR FEYNMAN DIAGRAM CALCULATIONS

C. A DOMINGUEZ

1994

$$\epsilon_{\alpha\beta\gamma\delta} = -\epsilon^{\alpha\beta\gamma\delta} \tag{1}$$

$$\epsilon_{\alpha\beta\gamma\delta}\epsilon^{\alpha\beta\gamma\delta} = -24 \tag{2}$$

$$\epsilon^{\alpha\beta\gamma\mu}\epsilon_{\alpha\beta\gamma\nu} = -6g^{\mu}_{\nu} \tag{3}$$

$$\epsilon^{\alpha\beta\mu\nu}\epsilon^{\rho\sigma}_{\alpha\beta} = -2(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \tag{4}$$

$$\begin{aligned}\epsilon^{\alpha\mu\nu\sigma}\epsilon^{\lambda\rho\tau}_{\alpha} = &-(g^{\mu\lambda}g^{\nu\rho}g^{\sigma\tau} + g^{\mu\rho}g^{\nu\tau}g^{\sigma\lambda} + g^{\mu\tau}g^{\nu\lambda}g^{\sigma\rho} \\ &- g^{\mu\lambda}g^{\nu\tau}g^{\sigma\rho} - g^{\nu\rho}g^{\mu\tau}g^{\sigma\lambda} - g^{\sigma\tau}g^{\mu\rho}g^{\nu\lambda})\end{aligned} \tag{5}$$

$$\epsilon^{\alpha\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma} = -det \begin{pmatrix} g^{\alpha\mu} & g^{\alpha\nu} & g^{\alpha\rho} & g^{\alpha\sigma} \\ g^{\beta\mu} & g^{\beta\nu} & g^{\beta\rho} & g^{\beta\sigma} \\ g^{\gamma\mu} & g^{\gamma\nu} & g^{\gamma\rho} & g^{\gamma\sigma} \\ g^{\delta\mu} & g^{\delta\nu} & g^{\delta\rho} & g^{\delta\sigma} \end{pmatrix} \tag{6}$$

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\, g_{\mu\nu} \tag{7}$$

$$\gamma^\mu\gamma_\mu = 4 \tag{8}$$

$$g^{\mu\nu}g_{\mu\nu} = 4 \tag{9}$$

$$g^{\mu\nu}\delta_{\mu\nu} = \begin{cases} -2 & \text{(Minkowski)} \\ +4 & \text{(Euclidean)} \end{cases} \tag{10}$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu \tag{11}$$

$$\gamma^\mu \gamma^\alpha \gamma^\beta \gamma_\mu = 4g^{\alpha\beta} \tag{12}$$

$$\gamma^\mu \gamma^\alpha \gamma^\beta \gamma^\rho \gamma_\mu = -2\gamma^\rho \gamma^\beta \gamma^\alpha \tag{13}$$

$$\gamma^\lambda \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\lambda = 2(\gamma^\sigma \gamma^\mu \gamma^\nu \gamma^\rho + \gamma^\rho \gamma^\nu \gamma^\mu \gamma^\sigma) \tag{14}$$

$$\sigma^{\mu\nu} = \frac{i}{2}\,[\gamma^\mu, \gamma^\nu] \tag{15}$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_5 = \frac{i}{4!}\,\epsilon_{\mu\nu\sigma\lambda}\,\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\lambda \tag{16}$$

$$\gamma^\mu\,\sigma^{\alpha\beta}\,\gamma_\mu = 0 \tag{17}$$

$$\gamma^\mu\,\sigma^{\alpha\beta}\,\gamma^\rho\,\gamma_\mu = 2\,\gamma^\rho\,\sigma^{\alpha\beta} \tag{18}$$

$$\gamma^\mu\,\gamma_5 + \gamma_5\,\gamma^\mu = 0 \tag{19}$$

$$\sigma_{\mu\nu}\,\sigma^{\mu\nu} = 12 \tag{20}$$

$$\gamma^\mu\gamma^\nu\gamma^\rho = g^{\mu\nu}\gamma^\rho - g^{\mu\rho}\gamma^\nu + g^{\nu\rho}\gamma^\mu + i\,\epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\sigma \tag{21}$$

$$\gamma_5\ \sigma^{\alpha\beta} = \frac{-i}{2}\ \epsilon^{\alpha\beta\mu\nu}\ \sigma_{\mu\nu} \tag{22}$$

$$\epsilon^{\lambda\mu\nu\sigma}\gamma_\mu\gamma_\nu\gamma_\sigma = 6i\gamma_5\gamma^\lambda \tag{23}$$

$$\sigma^{\mu\nu}\gamma^\lambda = ig^{\nu\lambda}\gamma^\mu - ig^{\mu\lambda}\gamma^\nu + \epsilon^{\mu\nu\lambda\sigma}\gamma_\sigma\gamma_5 \tag{24}$$

$$\gamma^\lambda\sigma^{\mu\nu} = -ig^{\nu\lambda}\gamma^\mu + ig^{\mu\lambda}\gamma^\nu + \epsilon^{\mu\nu\lambda\sigma}\gamma_\sigma\gamma_5 \tag{25}$$

$$\sigma^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}\gamma_\lambda\gamma_\sigma\gamma_5 \tag{26}$$

$$\sigma^{\mu\nu} = \frac{-i}{2}\epsilon^{\mu\nu\lambda\rho}\sigma_{\lambda\rho}\gamma_5 \tag{27}$$

$$\begin{aligned}\sigma^{\mu\nu}\sigma^{\alpha\beta} &= i\sigma^{\alpha\mu}g^{\beta\nu} - i\sigma^{\alpha\nu}g^{\beta\mu} + i\sigma^{\beta\nu}g^{\alpha\mu} \\ &- i\sigma^{\beta\mu}g^{\alpha\nu} + g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\nu}g^{\beta\mu} - i\epsilon^{\mu\nu\alpha\beta}\gamma_5\end{aligned} \tag{28}$$

$$(\gamma_5)^\dagger = \gamma_5 \tag{29}$$

$$(\gamma^\mu)^\dagger = \gamma_\mu = \begin{cases} \gamma^\mu & (\mu = 0) \\ -\gamma^\mu & (\mu \geq 1) \end{cases} \tag{30}$$

$$(\gamma_\mu)^\dagger = \gamma_0\gamma_\mu\gamma_0 \tag{31}$$

$$(\sigma_{\mu\nu})^{\dagger} = \gamma_0 \sigma_{\mu\nu} \gamma_0 \tag{32}$$

$$\bar{\Gamma} = \gamma^0 \Gamma^{\dagger} \gamma^0 \tag{33}$$

$$\bar{\gamma^{\mu}} = \gamma^{\mu} \tag{34}$$

$$\sigma^{\bar{\mu}\nu} = \sigma^{\mu\nu} \tag{35}$$

$$i\bar{\gamma}^5 = i\gamma^5 \tag{36}$$

$$C = i\,\gamma^2\,\gamma^0 = -C^{-1} = -C^{\dagger} = -C^T \tag{37}$$

$$C^2 = -1 \tag{38}$$

$$C^{-1}\,\gamma^{\mu}\,C = -\gamma^{\mu T} \tag{39}$$

$$C\,\gamma^{\mu T}\,C^{-1} = -\gamma^{\mu} \tag{40}$$

$$\gamma^{\mu}\,C = -C\gamma^{\mu T} \tag{41}$$

$$C\,\gamma^{\mu} = \gamma^{\mu T}\,C^{-1} = -\gamma^{\mu T}\,C \tag{42}$$

$$C^{-1}\,\gamma_5\,C = \gamma_5^T \tag{43}$$

$$C^{-1}\,\sigma_{\mu\nu}\,C = -\sigma_{\mu\nu}^{T} \tag{44}$$

$$C^{-1}\,\gamma_5\gamma_\mu\,C^{-1} = (\gamma_5\gamma_\mu)^{T} \tag{45}$$

$$\mathrm{Tr}\,(\gamma_\mu\gamma_\nu) = 4g_{\mu\nu} \tag{46}$$

$$\mathrm{Tr}\,(\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\lambda) = 4(g_{\mu\nu}g_{\sigma\lambda} + g_{\mu\lambda}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\lambda}) \tag{47}$$

$$\mathrm{Tr}\,(\gamma_5) = \mathrm{Tr}\,(\gamma_\mu\gamma_5) = \mathrm{Tr}\,(\gamma_\mu\gamma_\nu\gamma_5) = \mathrm{Tr}\,(\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_5) = 0 \tag{48}$$

$$\mathrm{Tr}\,(\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\lambda\gamma_5) = 4\;i\;\epsilon_{\mu\nu\sigma\lambda} \tag{49}$$

$$\begin{aligned}\mathrm{Tr}\,(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\lambda\gamma_\tau) &= 4i(g_{\mu\nu}\epsilon_{\rho\sigma\lambda\tau} - g_{\mu\rho}\epsilon_{\nu\sigma\lambda\tau} \\ &+ g_{\nu\rho}\epsilon_{\mu\sigma\lambda\tau} + g_{\sigma\lambda}\epsilon_{\mu\nu\rho\tau} - g_{\sigma\tau}\epsilon_{\mu\nu\rho\lambda} + g_{\lambda\tau}\epsilon_{\mu\nu\rho\sigma})\end{aligned} \tag{50}$$

$$\mathrm{Tr}\,(\sigma_{\kappa\lambda}\gamma_\mu\gamma_\rho) = 4\;i\;(g_{\kappa\rho}g_{\lambda\mu} - g_{\kappa\mu}g_{\lambda\rho}) \tag{51}$$

$$\mathrm{Tr}\,(\sigma_{\kappa\lambda}\gamma_\mu\gamma_\varphi\gamma_5) = -4\;\epsilon_{\kappa\lambda\mu\varphi} = 4\;\epsilon^{\kappa\lambda\mu\varphi} \tag{52}$$

$$\begin{aligned}\mathrm{Tr}\,(\gamma_a\gamma_b\gamma_c\gamma_d\gamma_e\gamma_f) &= 4(g_{ab}g_{cf}g_{de} - g_{ab}g_{ce}g_{df} - g_{ac}g_{bf}g_{de} \\ &+g_{ac}g_{be}g_{df} + g_{ad}g_{bf}g_{ce} - g_{ad}g_{be}g_{cf} + g_{bc}g_{af}g_{de} \\ &-g_{bc}g_{ae}g_{df} - g_{bd}g_{af}g_{ce} + g_{bd}g_{ae}g_{cf} - g_{cd}g_{af}g_{be} \\ &+ g_{cd}g_{ae}g_{bf} + g_{ad}g_{bc}g_{ef} - g_{ac}g_{bd}g_{ef} + g_{ab}g_{cd}g_{ef})\end{aligned} \tag{53}$$

$$\mathrm{Tr}\ \not{a}\ \not{b} = 4a \cdot b \tag{54}$$

$$\begin{aligned}\mathrm{Tr}\,(\not{a}_1\ \not{a}_2\ \not{a}_3\ \not{a}_4) = 4\,[&(a_1 \cdot a_2)(a_3 \cdot a_4) + (a_1 \cdot a_4)(a_2 \cdot a_3)\\ &- (a_1 \cdot a_3)(a_2 \cdot a_4)]\end{aligned} \tag{55}$$

$$\mathrm{Tr}\ (\gamma_\alpha\ \ \not{p}\,\gamma_\beta\ \ \not{p}\,\gamma_\beta\ \ \not{p}\,\sigma_{\mu\nu}\,\gamma_5) = -8\,p^2\,\epsilon_{\rho\alpha\mu\nu}\,p^\rho \tag{56}$$

$$\begin{aligned}&\mathrm{Tr}\ (\gamma_\alpha\ \ \not{p}'\gamma_\lambda\ \ \not{p}'\sigma_{\mu\nu}\,(\not{p}+m)\,\gamma_\lambda\,(\not{p}+m)\,\gamma_5)\\ &= 8m\left[p'^2\,\epsilon_{\rho\alpha\mu\nu}\,p^\rho - 2\,p'_\alpha\,\epsilon_{\rho\sigma\mu\mu}\,p^\rho\,p'^\sigma\right.\\ &\quad\left.+2\,p'_\mu\,\epsilon_{\rho\sigma\alpha\nu}\,p^\rho\,p'^\sigma - 2\,p'_\nu\,\epsilon_{\rho\sigma\alpha\nu}\,p^\rho\,p'^\sigma\right]\end{aligned} \tag{57}$$

$$\begin{aligned}&\mathrm{Tr}\ (\gamma_\alpha\ \ \not{p}'\gamma_\lambda\ \ \not{p}'\sigma_{\mu\nu}\,(\not{p}+m)\,\gamma_\rho\,(\not{p}+m)\,\gamma_5\,\sigma_{\lambda\rho})\\ &= 8\,i\,m\left(p^2\,\epsilon_{\rho\alpha\mu\nu}\,p'^\rho + 4\,p_\alpha\,\epsilon_{\rho\sigma\mu\nu}\,p^\rho\,p'^\sigma\right)\end{aligned} \tag{58}$$

$$\begin{aligned}&\mathrm{Tr}\ (\gamma_\alpha\ \ \not{p}'\gamma_\sigma\ \ \not{p}'\gamma_\rho\ \ \not{p}'\,\sigma_{\mu\nu}\,(\not{p}+m)\,\gamma_5\,\sigma_{\rho\sigma})\\ &= 8\,i\,m\,p'^2\,\epsilon_{\rho\alpha\mu\nu}\,p'^\rho\end{aligned} \tag{59}$$

$$\begin{aligned}(\lambda^a)^{\alpha\beta}\,(\lambda^a)^{\rho\lambda} \;=\;& 2(\delta_{\alpha\lambda}\,\delta_{\beta\rho} - \tfrac{1}{N_c}\delta_{\alpha\beta}\,\delta_{\rho\lambda})\\ &(a = 1, 2, \cdots 8)(\alpha, \beta = 1, 2, N_c)\end{aligned} \tag{60}$$

$$\epsilon_{\alpha\beta\gamma} \quad \epsilon_{\alpha\beta\gamma} = 6 \tag{61}$$

$$\mathrm{Tr}\,(\lambda^a \lambda^b) = 2\,\delta^{ab} \tag{62}$$

$$(\lambda^a)^{\alpha\beta} \ \left(\lambda^b\right)^{\beta\alpha} \ \delta^{ab} = 16 \tag{63}$$

$$\underbrace{q_i^\alpha\,(x)\;\bar{q}_j^\beta}(y) = i\;\delta^{\alpha\beta}\;S^q_{ij}\;(x-y) \tag{64}$$

$$S_F(p,m) = \int d^4x\; e^{ip\cdot x}\; S_F(x,m) \tag{65}$$

$$S_F(x,m) = \frac{1}{(2\pi)^4} \int d^4p\; e^{-ip\cdot x}\; S_F(p,m) \tag{66}$$

$$\begin{aligned} S_F^{\alpha\beta}(p,m) = {} & \frac{\delta^{\alpha\beta}}{\not{p} - m} - \frac{g}{2}\left(\frac{\lambda^a}{2}\right)^{\alpha\beta} \frac{G^a_{\kappa\lambda}}{(p^2-m^2)^2}\,[\,p_\varphi\;\epsilon^{\varphi\kappa\lambda\tau}\;\gamma_\tau\gamma_5 + m\;\sigma^{\kappa\lambda}] \\ & + \frac{g^2}{12}\;\delta^{\alpha\beta}\;G^2\;m\;\frac{p^2 + m\;\not{p}}{(p^2-m^2)^4} \qquad (\alpha,\beta = 1,2,N_c) \end{aligned} \tag{67}$$

$$\begin{aligned} S_F^{\alpha\beta}(p,m) = {} & \frac{\sim}{m \longrightarrow 0}\;\frac{\not{p}}{p^2}\;\delta^{\alpha\beta} + \frac{m}{p^2}\;\delta^{\alpha\beta} - \frac{g}{2}\left(\frac{\lambda^a}{2}\right)^{\alpha\beta} G^a_{\kappa\lambda}\;\frac{1}{p^4} \\ & \times \left[p_\varphi\;\epsilon^{\varphi\kappa\lambda\tau}\;\gamma_\tau\gamma_5 + m\;\sigma^{\kappa\lambda}\right] + \frac{\delta^{\alpha\beta}}{12}\;g^2\;G^2\;\frac{m}{p^6} \end{aligned} \tag{68}$$

$$S_F^{\alpha\beta}(x,m) = \frac{\sim}{m \longrightarrow 0} \frac{\delta^{\alpha\beta}}{2\pi^2} \frac{\gamma \cdot x}{x^4} + i\, m \frac{\delta^{\alpha\beta}}{4\pi^2} \frac{1}{x^2} + m^2 \frac{\delta^{\alpha\beta}}{8\pi^2} \frac{\gamma \cdot x}{x^2}$$

$$-\frac{1}{8\pi^2} \frac{x^\varphi}{x^2} \frac{g}{2} \left(\frac{\lambda^a}{2}\right)^{\alpha\beta} G^a_{\kappa\lambda}\, \epsilon^{\varphi\kappa\lambda\tau}\, \gamma_\tau\, \gamma_5$$

$$- i \frac{m}{16\pi^2} \frac{g}{2} \left(\frac{\lambda^a}{2}\right)^{\alpha\beta} G^a_{\kappa\lambda}\, \sigma^{\kappa\lambda}\, \Gamma(\underset{\uparrow \atop -2+D/2}{0}) + \; i m \frac{\delta^{\alpha\beta}}{3\pi^2 2^9}\, g^2\, G^2\, \Gamma(\underset{\uparrow \atop -3+D/2}{-1})\, x^2 \tag{69}$$

$$< 0|\bar{q}_i^\alpha(x) q_j^\beta(0)|0 > = \frac{1}{12}\, \delta^{\alpha\beta} \Big[\delta_{ij}(1 - \frac{x^2}{8}\, m_q^2)$$

$$+\frac{i}{4}\, m_q\, x^\mu (\gamma_\mu)_{ji} (1 - \frac{x^2}{12}\, m_q^2) \Big] < 0|\bar{q}q|0 >$$

$$+\frac{1}{12}\, \delta^{\alpha\beta} \Big[-\frac{i}{16}\, x^2\, (\delta_{ij} + \frac{i}{6}\, m_q\, x^\mu (\gamma_\mu)_{ji})$$

$$< 0|\bar{q} \sigma^{\mu\nu}\, i\, g\, \frac{\lambda^a}{2}\, G^a_{\mu\nu}\, q\, |0 >$$

$$+\frac{i}{288}\, x^2\, x^\mu (\gamma_\mu)_{ji}\, g^2\; < 0|\, \bar{q}\, \gamma^\nu\, \frac{\lambda^a}{2}\, q \sum_f \bar{q}^f\, \gamma_\nu\, \frac{\lambda^a}{2}\, q^f\, |0 > \Big] \tag{70}$$

$$< 0|\, G^a_{\alpha\beta}\, G^b_{\rho\lambda}\, |0 > = \frac{\delta^{ab}}{96}\, (g_{\alpha\rho}\, g_{\beta\lambda} - g_{\alpha\lambda}\, g_{\beta\rho})\; < 0|\, G^2\, |0 > \tag{71}$$

$$< 0|\, \bar{q}_i^\alpha(0)\, (i\, g\, \frac{\lambda^a}{2}\, G^a_{\mu\nu}\, (0)\, q_j\, (0))^\beta\, |0 >= \frac{\delta^{\alpha\beta}}{144}\; < 0|\, \bar{q}\, \sigma^{\mu\nu}\, i\, g\, \frac{\lambda^a}{2}\, G^a_{\mu\nu}\, q\, |0 > \tag{72}$$

$$\int \frac{d^D p}{(2\pi)^D} \frac{e^{-ip\cdot x}}{p^2} = i \frac{\Gamma(-1 + D/2)}{4\;\; \pi^{D/2}} \frac{1\, e^{-i\frac{\pi}{2}(D-4)}}{(x^2)^{D/2-1}} \tag{73}$$

$$\int d^D p \, \frac{e^{-ip\cdot x}}{(p^2)^n} = e^{-i\frac{\pi}{2}(D-1)} \, 2^{D-2n} \, \pi^{D/2} \, \frac{\Gamma(-n+D/2)}{\Gamma(n)} \, (x^2)^{n-D/2} \tag{74}$$

$$\int d^4x \, \frac{e^{iq\cdot x}}{x^2} = -\frac{4\pi^2 i}{q^2} \tag{75}$$

$$\int d^4x \, e^{iq\cdot x} \, \frac{x^\mu}{x^2} = \frac{8\pi^2}{(q^2)^2} \, q^\mu \tag{76}$$

$$\int d^4x \, e^{iq\cdot x} \, \frac{x^\mu x^\nu}{x^2} = \frac{8\pi^2 i}{(q^2)^3} \, (-q^2 g^{\mu\nu} + 4q^\mu q^\nu) \tag{77}$$

$$\int d^4x \, e^{iq\cdot x} \, \frac{x^\mu}{x^4} = \frac{2\pi^2}{q^2} \, q^\mu \tag{78}$$

$$\int d^4x \, e^{iq\cdot x} \, \frac{x^\mu x^\nu}{x^4} = \frac{2\pi^2 i}{(q^2)^2} \, (-q^2 g^{\mu\nu} + 2q^\mu q^\nu) \tag{79}$$

$$\int d^4x \, \frac{e^{iq\cdot x}}{(x^2)^n} = i \, \frac{\pi^2}{2^{2n-4}} \, \frac{(-)^n}{\Gamma(n)} \, \frac{1}{\Gamma(n-1)} \, \frac{\ln(-q^2)}{(q^2)^{2-n}} \quad (n > 1) \tag{80}$$

$$\int d^4x \, e^{iq\cdot x} \, \frac{x^\mu}{(x^2)^n} = \frac{\pi^2}{2^{2n-5}} \, \frac{(-)^n}{\Gamma(n)} \, q^\mu \, \frac{(n-2)}{\Gamma(n-1)} \, \frac{\ln(-q^2)}{(q^2)^{3-n}} \quad (n > 2) \tag{81}$$

$$\int d^4x \, e^{iq\cdot x} \, \frac{x^\mu x^\nu}{(x^2)^n} = i \, \frac{\pi^2}{2^{2n-5}} \, \frac{(-)^{n+1}}{\Gamma(n)} \, \frac{(n-2)}{\Gamma(n-1)} \, (q^2)^{n-3} \left(g^{\mu\nu} + 2(n-3) \frac{q^\mu q^\nu}{q^2} \right) \ln(-q^2) \quad (n > 2) \tag{82}$$

$$\int d^4x\, e^{iq\cdot x}\, \frac{x^\alpha x^\beta}{x^8} = -i\, \frac{\pi^2}{48}\, q^2 \left(g^{\alpha\beta} + \frac{2q^\alpha q^\beta}{q^2} \right) \ln(-q^2) \tag{83}$$

$$\begin{aligned} \int d^4x\, e^{iq\cdot x}\, \frac{x^\alpha x^\beta x^\rho}{(x^2)^n} &= \frac{\pi^2}{2^{2n-6}}\, \frac{(-)^{n+1}(n-2)(n-3)}{\Gamma(n)\Gamma(n-1)}\, (q^2)^{n-4} \\ &\times \left(q^\alpha g^{\beta\rho} + q^\beta g^{\alpha\rho} + q^\rho g^{\alpha\beta} + \frac{2(n-4)}{q^2} q^\alpha q^\beta q^\rho \right) \ln(-q^2) \end{aligned} \tag{84}$$

$$\begin{aligned} \int d^4x\, e^{iq\cdot x}\, \frac{x^\alpha x^\beta x^\rho}{x^{12}} &= -\frac{\pi^2}{2^7}\, \frac{1}{120}\, q^4 \Bigg(q^\alpha g^{\beta\rho} + q^\beta g^{\alpha\rho} \\ &+ q^\rho g^{\alpha\beta} + \frac{4}{q^2} q^\alpha q^\beta q^\rho \Bigg) \ln(-q^2) \end{aligned} \tag{85}$$

$$\frac{\partial}{\partial p_\beta} \left(\frac{1}{\not p} \right) = -\frac{1}{\not p}\, \gamma^\beta\, \frac{1}{\not p} \tag{86}$$

$$\frac{\partial^2}{(\partial p_\beta)^2} \left(\frac{1}{\not p} \right) = 2\, \frac{1}{\not p}\, \gamma^\beta\, \frac{1}{\not p}\, \gamma^\beta\, \frac{1}{\not p} \tag{87}$$

$$\frac{\partial}{\partial p_\beta} \left(\frac{1}{\not p - m} \right) = -\frac{1}{\not p - m}\, \gamma^\beta\, \frac{1}{\not p - m} \tag{88}$$

$$\psi(x) = \psi(0) + x_\alpha\, D_\alpha\, \psi(x)|_0 + \frac{1}{2}\, x_\alpha\, x_\beta\, D_\alpha\, D_\beta\, \psi(x)|_0 + \cdots \tag{89}$$

$$A_\mu(x) = \frac{1}{2}\, x_\alpha\, G_{\alpha\mu}(0) + \frac{1}{3}\, x_\alpha\, x_\beta\, D_\alpha\, G_{\beta\mu}(x)|_0$$
$$+ \frac{1}{4}\,\frac{1}{2!}\, x_\alpha\, x_\beta\, x_\lambda\, D_\alpha\, D_\beta\, G_{\lambda\mu}(x)|_0 + \cdots \tag{90}$$

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[ax + b(1-x)]^2} \tag{91}$$

$$\frac{1}{abc} = 2 \int_0^1 x\, dx \int_0^1 dy\, \frac{1}{[a(1-x) + bxy + cx(1-y)]^3} \tag{92}$$

$$\int \frac{d^D k}{(2\pi)^D}\, \frac{(k^2)^\alpha}{(k^2 - a^2)^\beta} = \frac{i(-a^2)^{\alpha-\beta+2}}{(4\pi)^2} \left(\frac{a^2}{4\pi}\right)^\epsilon$$
$$\frac{\Gamma(2+\alpha+\epsilon)\,\Gamma(\beta-\alpha-2-\epsilon)}{\Gamma(\beta) \qquad \Gamma(2+\epsilon)} \qquad (D = 4 + 2\epsilon)$$
$$\Gamma(-\epsilon) = -\frac{1}{\epsilon} - \gamma \tag{93}$$

$$I_0 \equiv \int d^D p \frac{1}{(p^2 \pm 2q\cdot p + M^2)^\alpha} = i\pi^{D/2} \frac{\Gamma(\alpha - D/2)}{\Gamma(\alpha)} (M^2 - q^2)^{D/2-\alpha} \tag{94}$$

$$\int d^D p \frac{p^\mu}{(p^2 \pm 2q\cdot p + M^2)^\alpha} = \mp q^\mu I_0 \tag{95}$$

$$\int d^Dp\frac{p^\mu p^\nu}{(p^2 \pm 2q\cdot p + M^2)^\alpha} = I_0\Bigg(q^\mu q^\nu + \frac{1}{2}g^{\mu\nu} \frac{\Gamma(\alpha - \frac{D}{2} - 1)}{\Gamma(\alpha - \frac{D}{2})}(M^2 - q^2)\Bigg) \tag{96}$$

$$\int d^Dp\frac{p^\mu p^\nu p^\lambda}{(p^2 \pm 2q\cdot p + M^2)^\alpha} = -I_0\Bigg(q^\mu q^\nu q^\lambda + \frac{1}{2}(g^{\mu\nu}q^\lambda + g^{\mu\lambda}q^\nu + g^{\nu\lambda}q^\mu)\frac{\Gamma(\alpha - \frac{D}{2} - 1)}{\Gamma(\alpha - \frac{D}{2})}(M^2 - q^2)\Bigg) \tag{97}$$

$$\int d^Dp\frac{p^\mu p^\nu p^\lambda p^\sigma}{(p^2 \pm 2q\cdot p + M^2)^\alpha} = I_0\Bigg(q^\mu q^\nu q^\lambda q^\sigma + \frac{1}{2}(g^{\mu\nu}q^\lambda q^\rho + g^{\mu\lambda}q^\nu q^\sigma + g^{\mu\sigma}q^\nu q^\lambda + g^{\nu\lambda}q^\mu q^\sigma + g^{\nu\sigma}q^\mu q^\lambda + g^{\lambda\sigma}q^\mu q^\nu)\frac{\Gamma(\alpha - \frac{D}{2} - 1)}{\Gamma(\alpha - \frac{D}{2})}(M^2 - q^2) + \frac{1}{4}(g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda})\frac{\Gamma(\alpha - \frac{D}{2} - 1)}{\Gamma(\alpha - \frac{D}{2})}(M^2 - q^2)\Bigg) \tag{98}$$

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2\ (k-q)^2} = \frac{-i}{(4\pi)^2}\Bigg(\ln(-q^2) - 2\Bigg) \tag{99}$$

$$\int \frac{d^4k}{(2\pi)^4}\frac{k^\mu}{k^2\ (k-q)^2} = \frac{-i}{(4\pi)^2}\frac{q^\mu}{2}\Bigg(\ln(-q^2) - 2\Bigg) \tag{100}$$

$$\int \frac{d^4k}{(2\pi)^4} \frac{k^\mu k^\nu}{k^2 \ (k-q)^2} = \frac{i}{(4\pi)^2} \Bigg(q^2 g^{\mu\nu} (\frac{\ln(-q^2)}{12} - \frac{2}{9}) + q^\mu q^\nu (-\frac{ln(-q^2)}{3} + \frac{13}{18}) \Bigg) \quad (101)$$

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 \ [(k-q)^2 - m^2]} = \frac{-i}{(4\pi)^2} \Bigg(\ln(-q^2) + \frac{m^2}{q^2} \ln(\frac{-m^2}{q^2}) + (1 - \frac{m^2}{q^2}) \ln(1 - \frac{m^2}{q^2}) - 2 \Bigg) \quad (102)$$

$$\int \frac{d^4k}{(2\pi)^4} \frac{k^\mu}{k^2 \ [(k-q)^2 - m^2]} = \frac{-i}{(4\pi)^2} \frac{1}{2} q^\mu \Bigg(\ln(-q^2) + \frac{m^2}{q^2} (2 - \frac{m^2}{q^2}) \times \ln(\frac{-m^2}{q^2}) + (1 - 2\frac{m^2}{q^2} + \frac{m^4}{q^4}) \ln(1 - \frac{m^2}{q^2}) + \frac{m^2}{q^2} - 2 \Bigg) \quad (103)$$

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - m^2] \ [(k-q)^2 - m^2]} = \frac{-i}{(4\pi)^2} \Bigg(\ln(\frac{m^2}{\nu^2}) + \sqrt{1 - 4\frac{m^2}{q^2}} \ \ln \frac{\sqrt{1 - 4\frac{m^2}{q^2}} + 1}{\sqrt{1 - 4\frac{m^2}{q^2}} - 1} - 2 \Bigg) \quad (104)$$

$$F_2(p_2^2, m_1^2, m_2^2) \equiv i \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m_1^2) \ [(p_2 - k)^2 - m_2^2]}$$

$$= + \frac{1}{16\pi^2} \int_0^1 dx \ \ln \left[m_1^2 x + m_2^2 (1-x) - p_2^2 x (1-x) \right] + \mathcal{O}(1/\epsilon) \quad (105)$$

$$F_3(q^2, p_1^2 = M^2,\ p_2^2 = M^2,\ \mu^2,\ M^2,\ m^2)$$

$$\equiv i \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - M^2)[(p_1 - k)^2 - \mu^2]\,[(p_2 - k)^2 - m^2]}$$

$$= \frac{1}{16\pi^2} \int_0^1 dx\, \frac{x}{\sqrt{-\Delta}} \ln\left\{ \frac{(\sqrt{-\Delta} - q^2x^2)^2 - (m^2 - \mu^2)^2 x^2}{(\sqrt{-\Delta} + q^2x^2)^2 - (m^2 - \mu^2)^2 x^2} \right\}$$

$$-\Delta = [-q^2x^2 + (m^2 - \mu^2)x]^2 - 4\, q^2x^2[M^2(1-x)^2 + \mu^2 x]$$

$$q^2 = (p_2 - p_1)^2 \tag{106}$$

$$F_3(q^2 = 0,\ p_1^2 = M^2,\ p_2^2 = M^2,\ \mu^2,\ M^2,\ m^2)$$

$$= \frac{1}{16\pi^2} \frac{1}{(\mu^2 - m^2)} \int_0^1 dx \ln\left\{ \frac{M^2x^2 + \mu^2(1-x)}{M^2x^2 + m^2(1-x)} \right\} \tag{107}$$