INTRODUCTION TO QUANTUM FIELD THEORY

(A *TOUR DE FORCE*)

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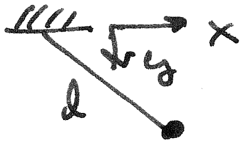
LECTURE I: Fast Review of Fundamental Concepts.

Classical Mechanics: $\vec{F} = \frac{d}{dt}\vec{p}\,;\ \vec{p} = m\frac{d\vec{r}}{dt}$ (I.1)

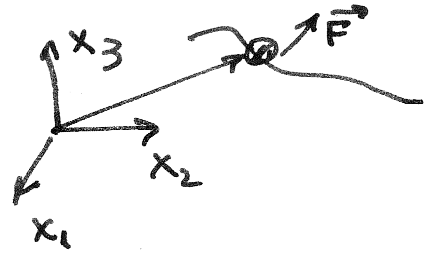
Constraints, e.g.



$x^2(t) + y^2(t) = l^2$
$= \text{constant}$

(*) Replace cartesian coordinates with Generalized Coordinates $q_i(t)$ $(i = 1, \dots N)$ all independent of each other.

(*) Instead of Forces use Energies



$\vec{x} = (x_1, x_2, x_3)$ $\quad (\dot{x}_i \equiv \frac{d}{dt} x_i)$

$$F_i = \frac{d}{dt} p_i = \frac{d}{dt}(m\dot{x}_i) = \frac{d}{dt}\frac{\partial}{\partial \dot{x}_i}\underbrace{\left(\tfrac{1}{2}m\dot{x}_i^2\right)}_{T}$$

$$= \frac{d}{dt}\frac{\partial}{\partial \dot{x}_i} T$$

Kinetic energy

In cartesian coordinates $\frac{\partial}{\partial x_i} T = 0$

Conservative systems
$$\begin{cases} \frac{\partial}{\partial \dot{x}_i} V = 0 \\ F_i = -\frac{\partial V}{\partial x_i} \end{cases}$$

V: potential energy

$$\frac{\partial}{\partial x_i}(T - V) = \frac{d}{dt}\frac{\partial}{\partial \dot{x}_i}\underbrace{(T - V)}_{L \text{ (Lagrangian)}}$$

Lagrange's Equations
$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0} \quad \text{(I.2)}$$

$$x_i(t) \Longrightarrow q_i(t) \qquad i = 1, 2, \dots N$$

N: number of "degrees of freedom".

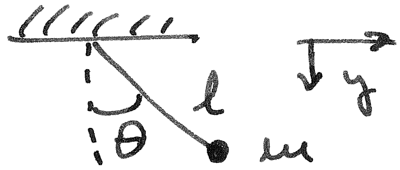
Generalized momentum: $p_i = \dfrac{\partial L}{\partial \dot{q}_i}$

$$\text{(I.3)} \quad \boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0} \Rightarrow \boxed{\frac{d}{dt} p_i - \frac{\partial L}{\partial q_i} = 0} \quad \text{(I.4)}$$

If $\dfrac{\partial L}{\partial q^*} = 0 \Rightarrow \dfrac{d}{dt} p^* = 0 \therefore p^* =$ constant of motion.

CONFIGURATION SPACE

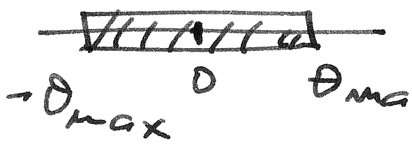
The space generated by the generalized coordinates. Example



$$x^2(t) + y^2(t) = l^2$$

$$\Rightarrow q_1 \equiv \theta(t) \text{ generalized coordinate}$$

Configuration space:



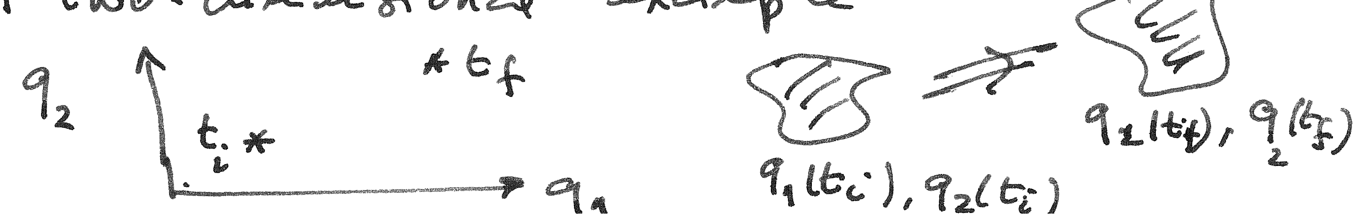
A LINE

ACTION

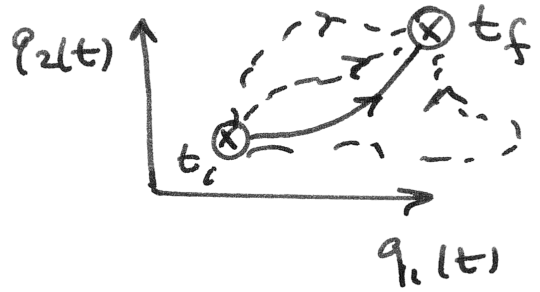
$$S = \int_{t_1}^{t_2} L(q_i(t), \dot{q}_i(t), t)\, dt \quad \text{(I.5)}$$

Dim [S] = Energy * Time (e.g. Joule * sec)

A two-dimensional example



Question: which path? ONLY ONE. Which one?



The one that "minimizes" S

Hamilton Principle

$$\boxed{\delta S = 0} \quad (\text{I}.6) \text{ with } t_1 \,\&\, t_2 \text{ fixed}$$

$$\delta S = \int_{t_1}^{t_2} \delta L \, dt = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial (\dot q_i)} \delta \dot q_i \right] dt$$

$$\int_{t_1}^{t_2} dt \frac{\partial L}{\partial \dot q_i} \delta \dot q_i = \int_{t_1}^{t_2} dt \frac{\partial L}{\partial \dot q_i} \frac{d}{dt} \delta q_i = \underbrace{\left[\delta q_i \frac{\partial L}{\partial \dot q_i} \right]_{t_1}^{t_2}}_{\delta q_i(t_1) = \delta q_i(t_2) = 0} - \int_{t_1}^{t_2} dt \, \delta q_i \frac{d}{dt} \left(\frac{\partial L}{\partial \dot q_i} \right)$$

$$\delta S = \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot q_i} \right) \right] \delta q_i$$

Using (I.6) & since δq_i is arbitrary,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot q_i} \right) - \frac{\partial L}{\partial q_i} = 0$$

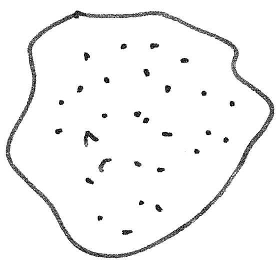
which are Lagrange's eqns (I.3) - (I.4).

Hamiltonian:

$$H = \sum_{i=1}^{N} (p_i \dot q_i - L) \quad (\text{I}.7)$$

In most cases $H = T + V$, i.e. the total energy of the system.

FIELDS



$q_i(t)$, $i = 1, 2 \ldots N$.

For large N, e.g. $N \simeq N_A \approx 6 \times 10^{23}$

treat system as a "continuum"

$q_{i}(t)$

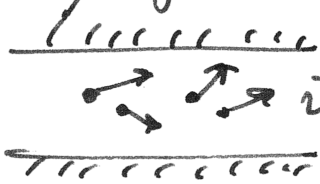
$i = 1, 2, \ldots N \to \infty$ & non-denumerable, meaning i becomes a continuous variable, i.e. $\vec{x} = (x_1, x_2, x_3)$

$$q_i(t) \Longrightarrow q(\vec{x}, t) \equiv \underbrace{\varphi(\vec{x}, t)}_{\text{FIELD.}} \quad \text{(I.8)}$$

FIELD: function of $\vec{x}$ & t describing a continuum !

Example: i) Deformations of a non-rigid body

ii) Velocity field in fluid flow:



$\vec{v} = \vec{v}(\vec{x}, t)$

LAGRANGIAN DENSITY

$$L(q_i(t), \dot{q}_i(t)) \Longrightarrow \mathcal{L}\left(\varphi(\vec{x}, t), \vec{\nabla}\varphi(\vec{x}, t), \frac{\partial}{\partial t}\varphi(\vec{x}, t)\right)$$

$$\boxed{L \Rightarrow \int d^3x \, \mathcal{L}} \quad (\text{I}.9)$$

Relativistic Notation:

$$x^\mu = (x^0 = x_0 = ct; \vec{x}); \quad x_\mu = (ct, -\vec{x})$$

$$x_\mu = g_{\mu\nu} x^\nu; \quad g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g^{\mu\nu}$$

Metric Tensor

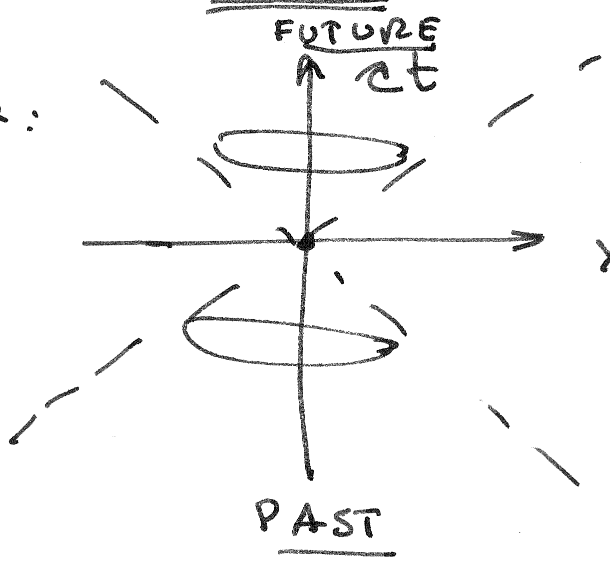
Minkowski space

$(\vec{x}, t)$ 4-dimensional space

Distance: $x^\mu x_\mu = x^\mu g_{\mu\nu} x^\nu = x_0^2 - \vec{x}^2$

Convention: $c = 1$

Light Cone:



Derivatives

$$\frac{\partial}{\partial x_\mu} \Rightarrow \partial^\mu$$

$$\frac{\partial}{\partial x^\mu} \Rightarrow \partial_\mu$$

$$\mathcal{L} = \mathcal{L}(\varphi(x), \partial^\mu \varphi(x)) \quad (\text{I}.10)$$

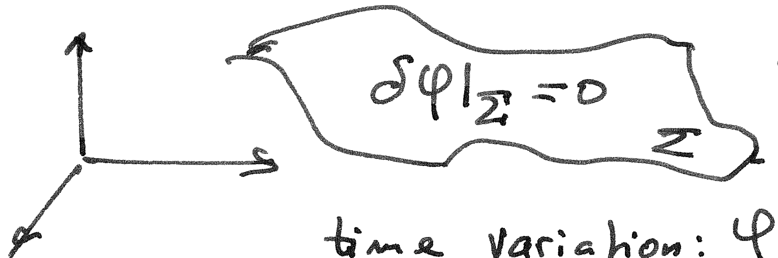
$$S = \int d^4x \, \mathcal{L}(\varphi(x), \partial^\mu \varphi(x)) \quad (\text{I}.11)$$

Hamilton's principle: $\delta S = 0$

leads to Lagrange's equations for $\varphi(x)$

δS : a variation in (1) time & (2) space
i.e. on a surface (3-dimensional)

$\delta\varphi(t_1) = \delta\varphi(t_2) = 0$
fixed times.



$$S = \int_{t_1}^{t_2} dt \int_\Sigma d^3x\, \mathcal{L}(\varphi, \partial_\mu\varphi)$$

time variation: $\varphi \to \varphi' = \varphi + \delta_0\varphi$

$$\delta S = \int_{t_1}^{t_2} dt \int_\Sigma d^3x \left[\frac{\partial \mathcal{L}}{\partial\varphi(x)} \delta_0\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)} \delta_0(\partial_\mu\varphi) \right]$$

$$\int_{t_1}^{t_2} dt \int_\Sigma d^3x \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)} \delta_0(\partial_\mu\varphi) =$$

$$= \int_{t_1}^{t_2} dt \int_\Sigma d^3x \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)} \partial_\mu(\delta_0\varphi)$$

$$= \underbrace{\left[\int_\Sigma \delta_0\varphi \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)} \right]_{t_1}^{t_2}}_{=0} - \int_{t_1}^{t_2} dt \int_\Sigma d^3x\, \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)} \delta_0\varphi$$

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial\varphi(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)} \right] \delta_0\varphi = 0 \quad (\text{I.12})$$

$$\Rightarrow \quad \boxed{\frac{\partial \mathcal{L}}{\partial\varphi(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)} = 0} \quad (\text{I.13})$$

Field Momentum:

$$\boxed{\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi)}} \qquad (\mathrm{I}.14)$$

(to mimic $p_i \equiv \partial L / \partial \dot{q}_i$)

Hamiltonian Density:

$$H = \sum_i p_i \dot{q}_i - L \implies \boxed{\mathcal{H} = \sum_i \pi_i \frac{\partial \varphi_i}{\partial t} - \mathcal{L}} \qquad (\mathrm{I}.15)$$

— x —

Electromagnetic Field (Classical)

The only field in Nature that exists at the classical (i.e. non-quantum) level. Quantum fields associated with leptons (e, μ, τ) & quarks (u, d, s, c, b, t) do not have a classical limit. They simply do not exist at the classical level!

E.M. Field $\implies$ Maxwell

$$\begin{cases} \vec{E}(x) \equiv \vec{E}(\vec{x}, t) \\ \vec{B}(x) \equiv \vec{B}(\vec{x}, t) \end{cases} \qquad (\mathrm{I}.16)$$

$$\left.\begin{aligned} &\vec{\nabla}\cdot\vec{E}(\vec{x},t) = \rho(\vec{x},t) \\ &\vec{\nabla}\times\vec{B}(\vec{x},t) - \frac{\partial}{\partial t}\vec{E}(\vec{x},t) = \vec{J}(\vec{x},t) \\ &\vec{\nabla}\cdot\vec{B}(\vec{x},t) = 0 \\ &\vec{\nabla}\times\vec{E}(\vec{x},t) + \frac{\partial}{\partial t}\vec{B}(\vec{x},t) = 0 \end{aligned}\right\} \quad (\text{I}.17)$$

Potentials: $\Phi(\vec{x},t)$ & $\vec{A}(\vec{x},t) \Rightarrow A^\mu(x)$ (I.18)

$$A^\mu(x) \Rightarrow [\Phi(x), \vec{A}(x)] \quad (\text{I}.19)$$

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t} \quad ; \quad \vec{B} = \vec{\nabla}\times\vec{A} \quad (\text{I}.20)$$

EM Tensor: $\boxed{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = -F^{\nu\mu}}$ (I.21)

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} \quad (\text{I}.22)$$

SHOW: $\boxed{\mathcal{L}_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}}$ (I.23)

in the absence of sources, i.e. in free space ($\rho=0, \vec{J}=0$)... this $\mathcal{L}_{EM}$ satisfies

$$\boxed{\partial^\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial\mathcal{L}}{\partial(A_\nu)} = 0} \quad (\text{I}.24)$$

<u>SHOW:</u>

$$\partial_\mu F^{\mu\nu}(x) = J^\nu(x) \qquad (\text{I}.25)$$

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \qquad (\text{I}.26)$$

Show that Eq. (I.25) leads to the two inhomogeneous Maxwell eqns. and (I.26) leads to the other two (homogeneous) eqns. –

<u>HINT:</u> $\dfrac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)}$ leads to 4 identical terms

$$-\frac{1}{4}\, g_{\lambda\mu}\, g_{\nu\sigma} \left\{ \delta^\mu_\beta\, \delta^\sigma_\alpha\, F^{\lambda\nu};\ \delta^\sigma_\beta\, \delta^\mu_\alpha\, F^{\lambda\nu};\ \delta^\lambda_\beta\, \delta^\nu_\alpha\, F^{\mu\nu};\ \delta^\nu_\beta\, \delta^\lambda_\alpha\, F^{\mu\nu} \right\} \quad (\text{I}.27)$$

hence

$$\frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)} = -g_{\lambda\beta}\, g_{\nu\alpha}\, F^{\lambda\nu} = -F_{\beta\alpha} = F_{\alpha\beta} \qquad (\text{I}.28)$$

We shall return later to the EM field to discuss its quantization.

Noether's Theorem

For every continuous symmetry of a Lagrangian there is a conserved current ($\partial^\mu J_\mu (x) = 0$) built from the fields in the Lagrangian.

Proof Assume $\mathcal{L} = \mathcal{L}(\varphi(x), \partial_\mu \varphi(x))$

trivially generalized to an $\mathcal{L}$ depending on more than one field.

Perform a transformation of the field

$$\varphi(x) \longrightarrow \varphi'(x) = \varphi(x) + \delta\varphi(x) \quad (\text{I}.29)$$

e.g. $\varphi(x) \to \varphi'(x) = e^{i\vartheta}\varphi(x) \simeq (1 + i\vartheta + \dots)\varphi(x)$

The change in $\mathcal{L}$ is

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta(\partial_\mu\varphi) \quad (\text{I}.30)$$

use Lagrange's eqns $\dfrac{\partial\mathcal{L}}{\partial\varphi} = \partial_\mu \dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}$

to write

$$\delta\mathcal{L} = \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\mu\delta\varphi$$

$$\boxed{\delta\mathcal{L} = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi\right]} \quad (\text{I}.31)$$

If the transformation (I.29) leaves the Lagrangian invariant, i.e.

if $\delta\mathcal{L} = 0$ then $J_\mu \equiv \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}$ (I.32)

is conserved, i.e.

$$\underline{\partial^\mu J_\mu = 0} \qquad \text{(I.33)}$$

<u>Example</u>:

Classical Electrodynamics in Vacuum

$$\mathcal{L}_{CED} = -\frac{1}{4} F_{\mu\nu}(x)\, F^{\mu\nu}(x)$$
$$= -\frac{1}{4} \left(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right)$$
$$\times \left(\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)\right)$$

$$A_\mu(x) \Rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x)$$

$$\Rightarrow \mathcal{L}'_{CED} = \mathcal{L}_{CED}$$

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{4}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)$$

$$\partial_\mu J^\mu = -\frac{1}{4}\Big(\underbrace{\partial_\mu \partial^\mu A^\nu}_{\substack{=0 \\ \text{from equation} \\ \text{of motion: } \Box^2 A^\nu = 0 \\ \text{in free space}}} - \partial^\nu \underbrace{\partial_\mu A^\mu}_{\substack{=0 \\ \text{(Lorentz)}}}\Big) = 0.$$

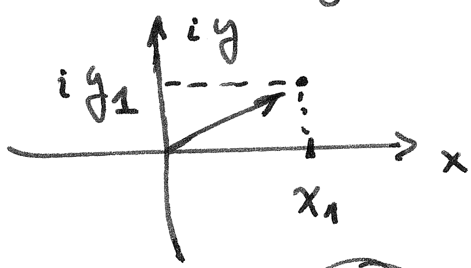
(Non-relativistic) QUANTUM MECHANICS

Schrödinger equation: (Atomic, Molecular, Physics & CHEMISTRY)

(I.34) $$\sqrt{-1}\,\hbar \frac{\partial \Psi(\vec{x},t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi(\vec{x},t) + V\Psi(\vec{x},t)$$

— x —

Complex numbers: $z = (z_1, z_2)$ a pair of numbers, e.g. $z_1 = x$, $z_2 = \sqrt{-1}\,y \equiv iy$



$$\boxed{i^2 = -1} \quad \text{(I.35)}$$

$$\boxed{e^{i\pi} = -1} \quad \text{(I.36)}$$

i RELATES e WITH π !

profound meaning.

— x —

$\Psi(\vec{x},t)$ a complex function

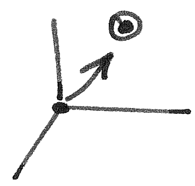
$$|\Psi(\vec{x},t)|^2 = \Psi(\vec{x},t)\,\Psi^*(\vec{x},t)$$

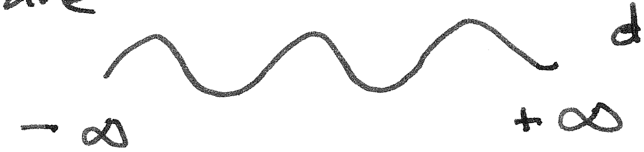
probability of finding the "particle" described by $\Psi(\vec{x},t)$ at point $\vec{x}$ & at time t.

"WAVE-PARTICLE DUALITY"

A nonsensical statement

CLASSICAL UNIVERSE

"particle":  localized point object (any object whose dimensions are negligible in the problem at hand)

"wave"  delocalized object

Opposite concepts ONLY in the classical domain. At the quantum level, this concept/definition breaks down. We should call an e^- neither a particle nor a wave, but something else, e.g. "dualon". Explanation of this conundrum is not be found in QUANTUM MECHANICS but only in QUANTUM FIEL THEORY.

e.g. $\underset{\sim}{e^-}$. What is the electron?
There is a quantum field describing the electron (Dirac field). Its quantum excitations (of the field) manifest themselves as particles.

Schrödinger current:
Assume $V=0$ & write the eqn. and its complex conjugate

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi \quad \text{\& multiply by } \Psi^*$$

$$-i\hbar \frac{\partial \Psi^*}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi^* \quad \text{\& multiply by } \Psi$$

---

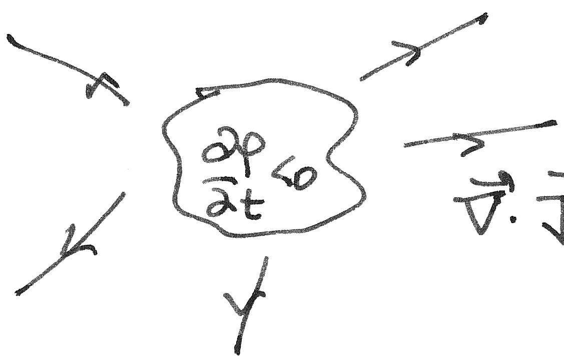
subtracting gives

$$\frac{\partial}{\partial t}\left(\Psi^*(\bar{x},t)\, \Psi(\bar{x},t)\right) = \frac{i\hbar}{2m}\left(\Psi^* \nabla^2 \Psi - \Psi \nabla^2 \Psi^*\right)$$

$$\boxed{\frac{\partial}{\partial t} |\Psi(\bar{x},t)|^2 \equiv \frac{\partial \rho(\vec{x},t)}{\partial t} = -\vec{\nabla}\cdot \vec{J}_S(\bar{x},t)} \quad (\text{I}.37)$$

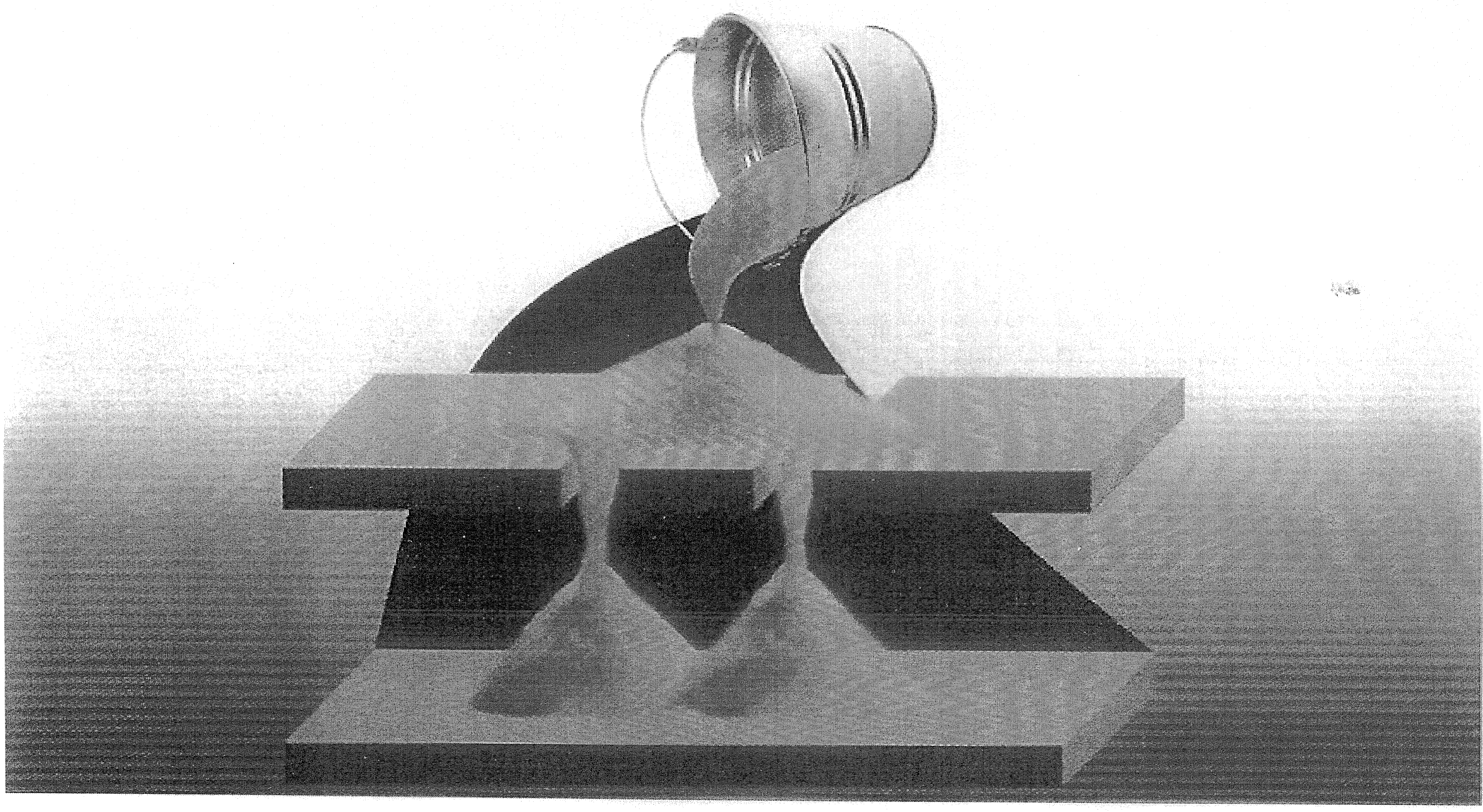
$$\boxed{\vec{J}_S(\bar{x},t) \equiv \frac{i\hbar}{2m}\left(\Psi \vec{\nabla}\Psi^* - \Psi^* \vec{\nabla}\Psi\right)} \quad (\text{I}.38)$$

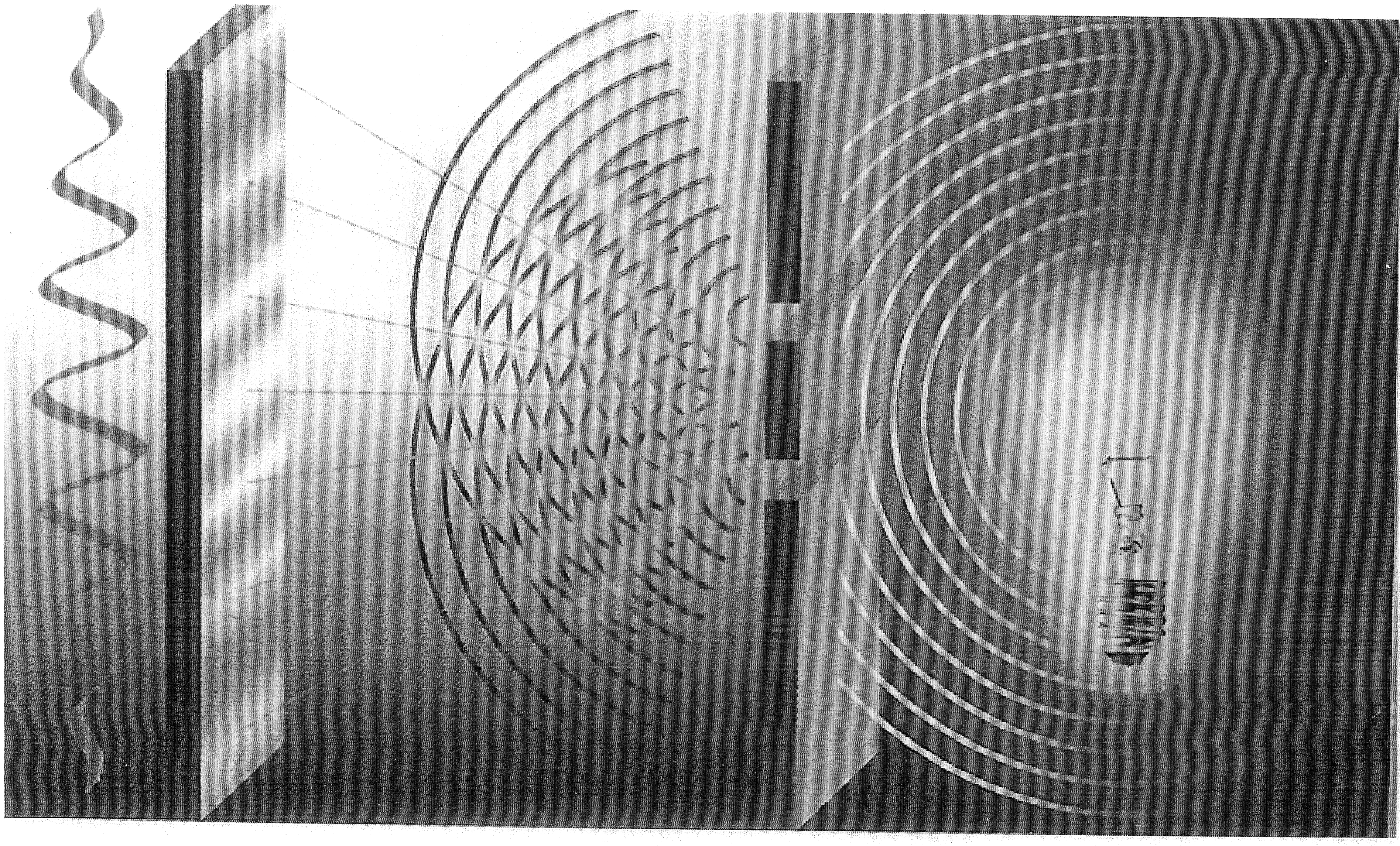
If $\frac{\partial \rho}{\partial t} < 0$ in a volume then $\vec{\nabla}\cdot\vec{J}_S > 0$

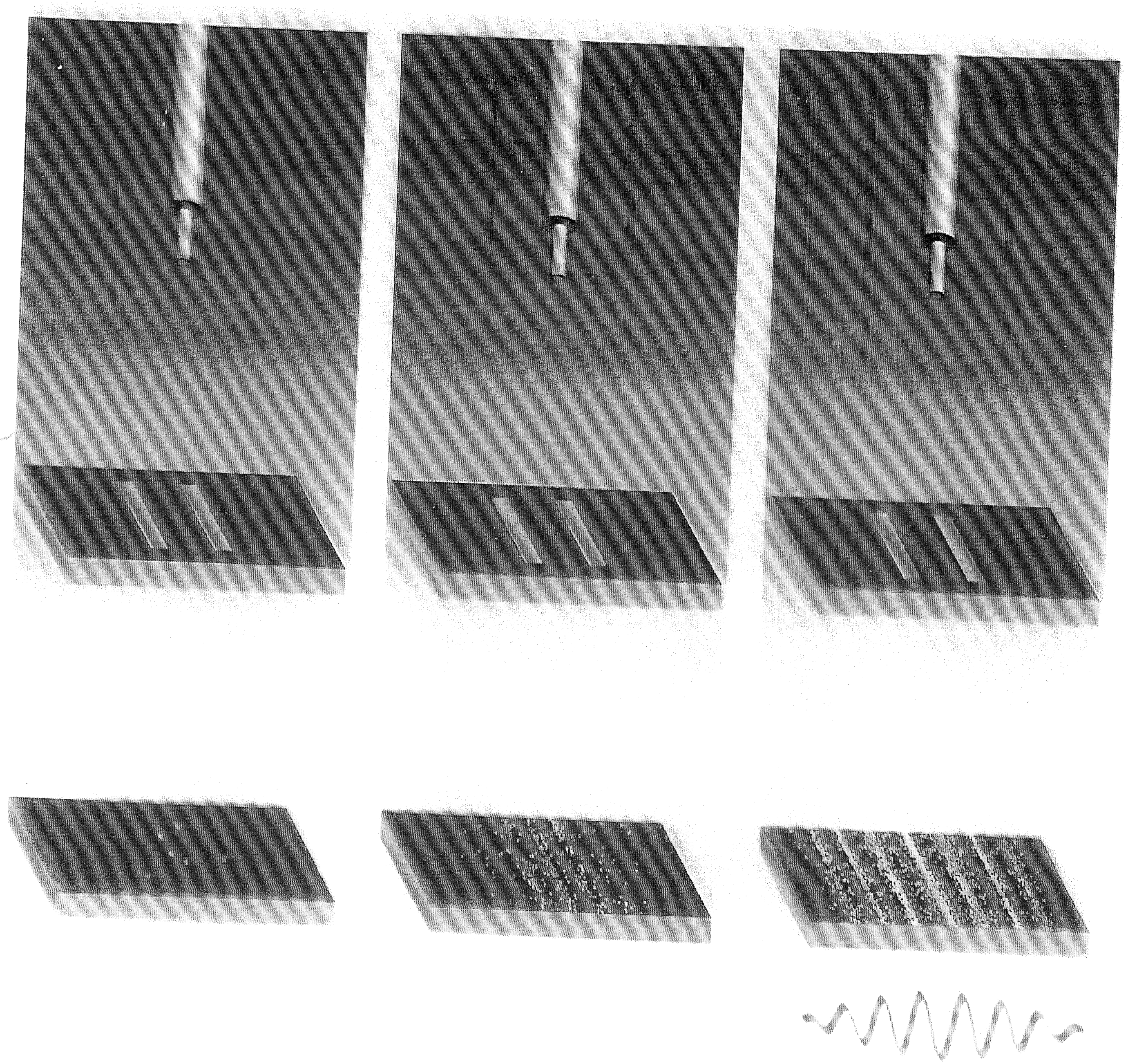


$\vec{\nabla}\cdot\vec{J} > 0$

<u>SLIDES</u>: classical & quantum interference.







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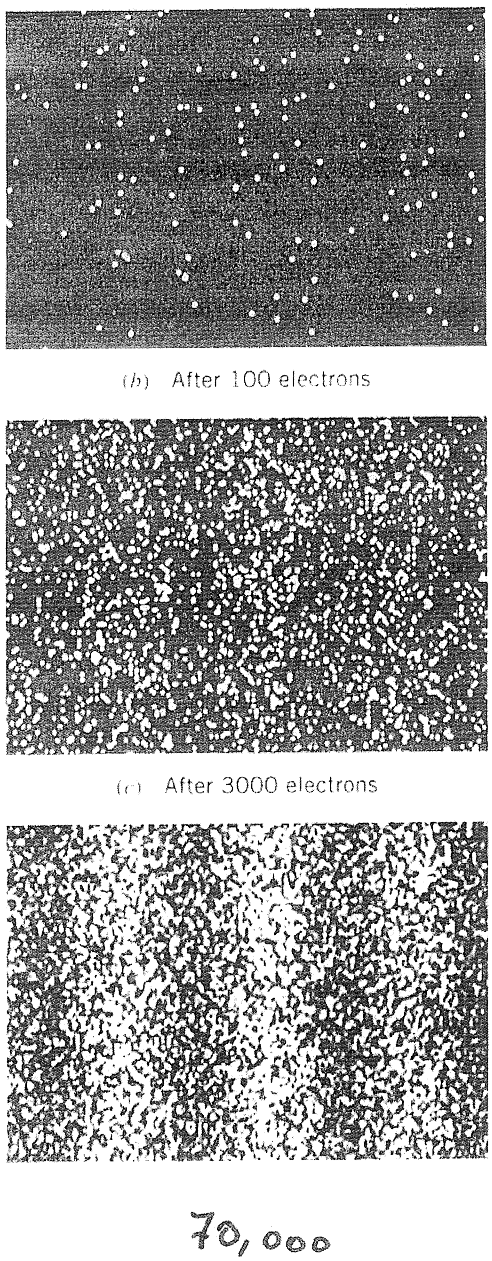


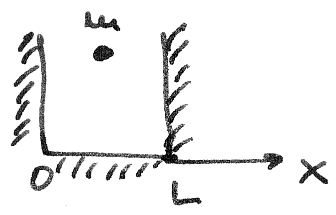
(b) After 100 electrons

(c) After 3000 electrons

70,000

TUTORIAL I

1) Solve the Schrödinger equation for a particle in a one-dimensional box (line) of length L, bounded by infinite walls. Consider both $E<0$ and $E>0$. Consider the time-independent case.



$$V(x) = \begin{cases} \infty & x<0 \\ \infty & x>L \\ 0 & 0<x<L \end{cases}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\,\psi(x)$$

a) $\underline{E<0}$: there are no solutions.

b) $\underline{E>0}$

$$\psi(x) \equiv u(x) \qquad k^2 \equiv \frac{2mE}{\hbar^2} > 0$$

$$\frac{d^2u(x)}{dx^2} + k^2 u(x) = 0$$

$$u(x)\big|_{x=0} = 0 \qquad u(x)\big|_{x=L} = 0$$

$$u(x) = A\sin kx + B\cos kx$$

but $B=0$ & $kL = n\pi$

$$\therefore \boxed{u(x) = A\sin\left(\frac{n\pi}{L}x\right)}$$

$$E_n = \frac{\hbar^2}{2m}k_n^2 = \frac{\hbar^2}{2m}\frac{n^2\pi^2}{L^2}$$

$$\boxed{E_n = \frac{\hbar^2}{2m}\frac{n^2\pi^2}{L^2}} \qquad E_n \propto n^2$$