etc

4 - - - - - - - $n=2$

1 - - - - - - $n=1$

$$\Delta E = \frac{\hbar^2}{2m}\frac{\pi^2}{L^2}\left((n+1)^2 - n^2\right)$$

$$\boxed{\Delta E = \frac{\hbar^2}{2m}\frac{\pi^2}{L^2}(2n+1)}$$

$$\lim_{\hbar \to 0} \Delta E = 0$$

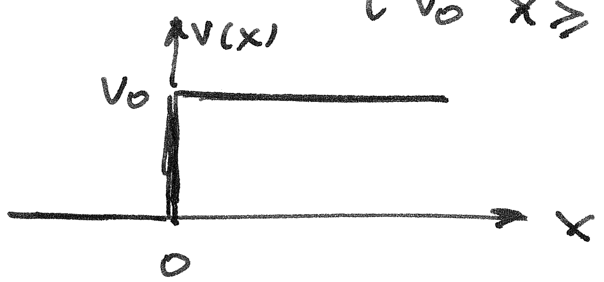
Classical limit: continuum.

—— x ——

2) Solve the time independent Schrödinger equation for a particle scattering off a step-function potential

$$V(x) = \begin{cases} 0 & x<0 \\ V_0 & x \geq 0 \end{cases}$$

. Consider $E < V_0$ & $E > V_0$.



$$\frac{d^2u(x)}{dx^2} + \frac{2m}{\hbar^2}\left[E - V(x)\right]u(x) = 0$$

$$k^2 = \frac{2mE}{\hbar^2} \quad ; \quad q^2 = \frac{2m(E-V_0)}{\hbar^2}$$

a) $\underline{x<0}$

$$u(x) = A e^{ik\cdot x} + B e^{-ik\cdot x}$$

$$= e^{ik\cdot x} + R e^{-ik\cdot x}$$

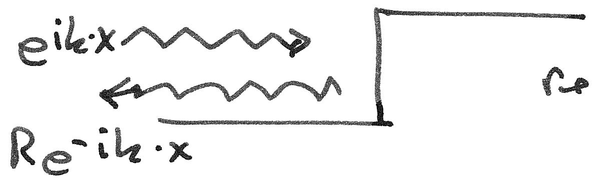
where $R \equiv B/A$ & $A = 1$.

$$j_x = \frac{i\hbar}{2m}\left(u \frac{d}{dx} u^* - u^* \frac{d}{dx} u\right)$$

$$\boxed{j_x = \frac{\hbar k}{m}\left(1 - |R|^2\right)}$$

where $|R| \neq 0$ signals the presence of the potential (step) @ $x \geq 0$.

In time:

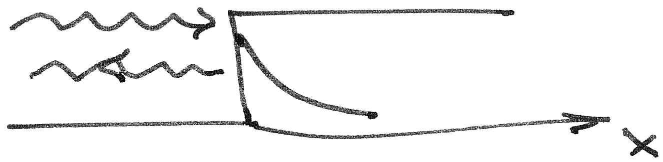


reflection @ $x = 0$.

(b) $\underline{\underline{x > 0}}$ if $E < V_0$, $q^2 < 0$ $\therefore$ $q = i\kappa$

$$\kappa = \frac{2m(V_0 - E)}{\hbar^2}$$

$$u(x) \propto e^{-\kappa x}$$ exponentially suppressed.



For $\underline{E > V_0}$ q is real

$$\begin{cases} u(x) = T\, e^{iqx} & x \geq 0 \\ j_x = \frac{\hbar q}{m} |T|^2 \end{cases}$$

Boundary conditions: continuity of wave function & its first derivative at the boundary.

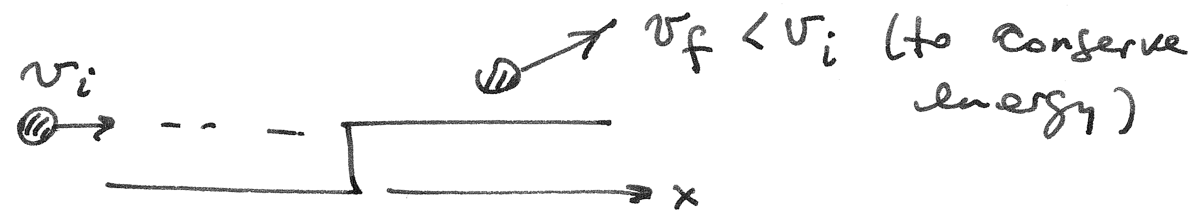
$$\begin{cases} u(x)\big|_{0^-} = u(x)\big|_{0^+} & (a) \\ \frac{du}{dx}\big|_{0^-} = \frac{du}{dx}\big|_{0^+} & (b) \end{cases}$$

(a): $1 + R = T$

(b): $ik(1 - R) = iqT$

$$R = \frac{k - q}{k + q}; \quad T = \frac{2k}{k + q}$$

Classical Case ($\hbar \to 0$)



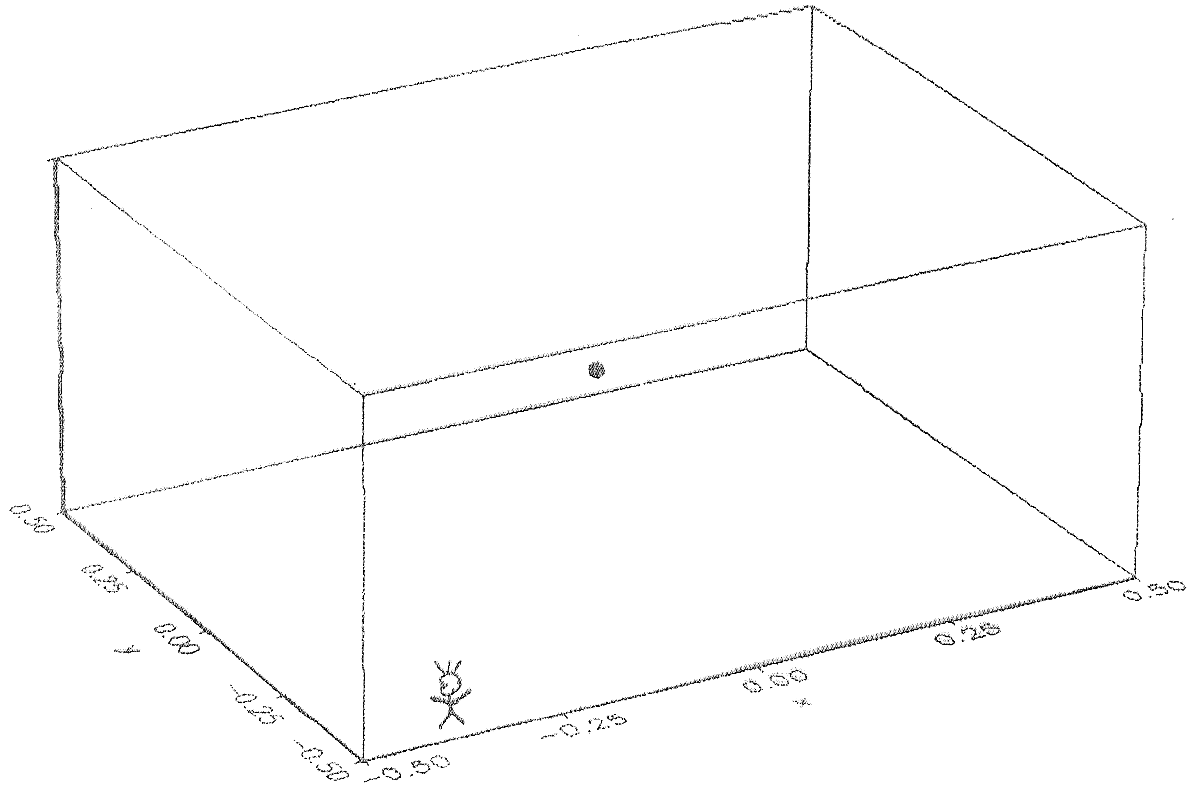
There is NO REFLECTION classically, i.e. $R = 0$.

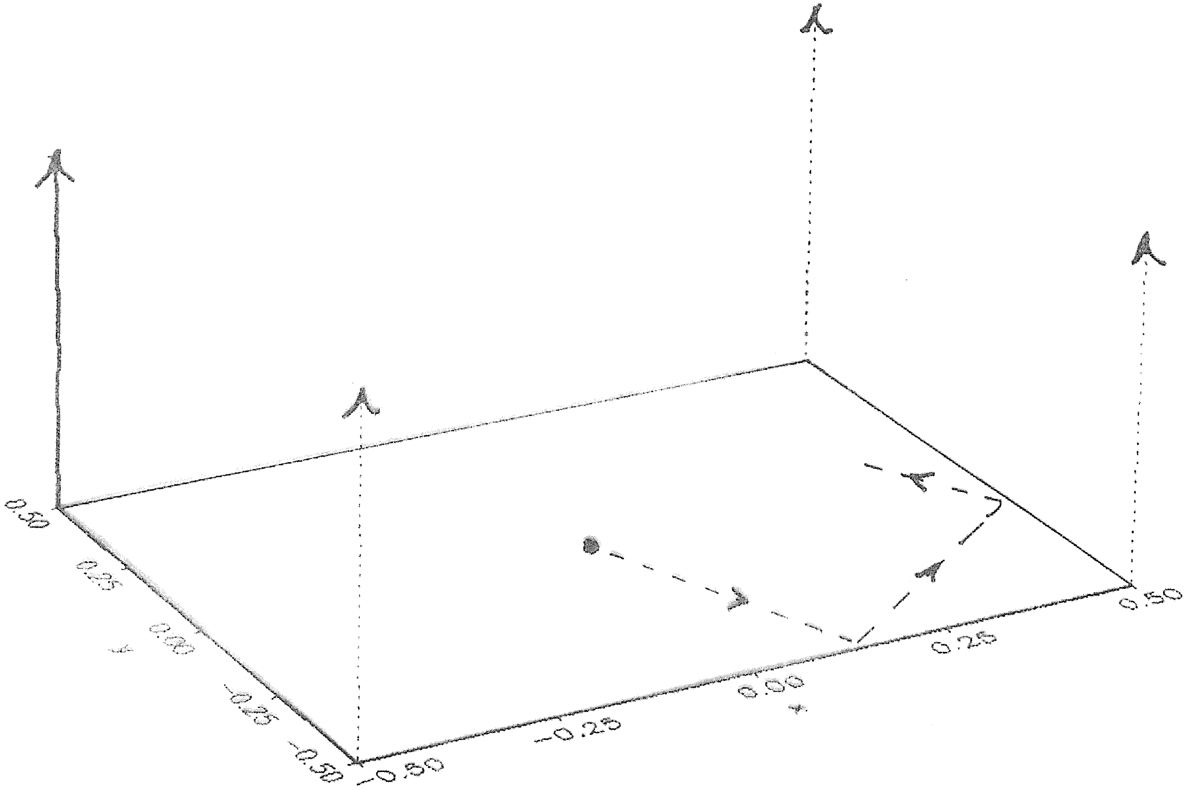
Quantum case ($\hbar \neq 0$)

$$R \neq 0$$

Hence, it "seems" as if the particle is behaving as a wave. BUT, it is neither a particle nor a wave. These are macroscopic concepts based on brain software from macroscopic experience. At the quantum level, in non-relativistic Quantum Mechanics, there is no solution to this puzzle. Must wait for Quantum Field Theory to define a particle.

3) Solve Schrödinger's eqn. for a particle inside a rectangular box (2 dimensions) of length L (free particle in the box).


0.50
0.25
0.00
y
−0.25
−0.50
−0.50
−0.25
0.00
x
0.25
0.50



0.50
0.25
0.00
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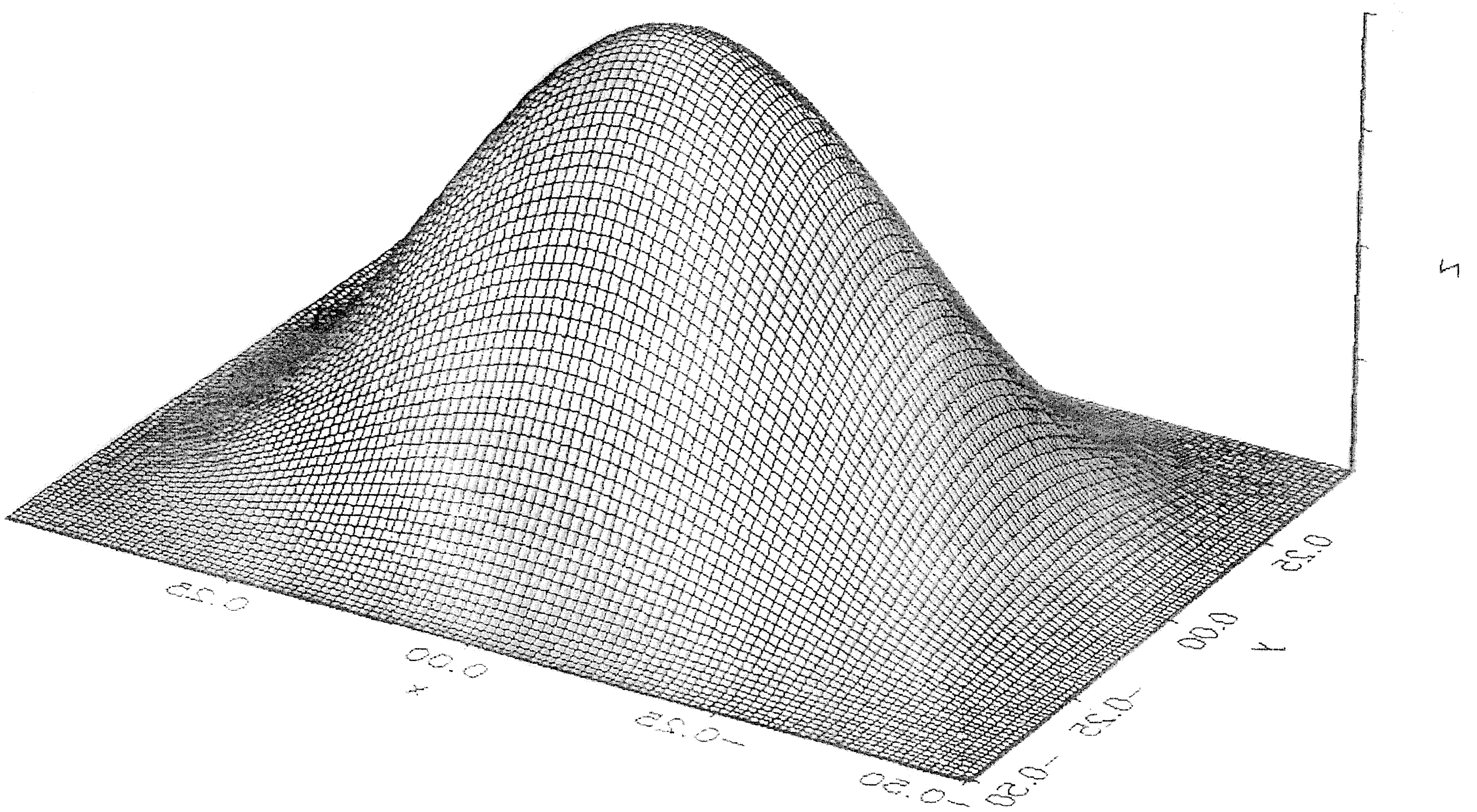
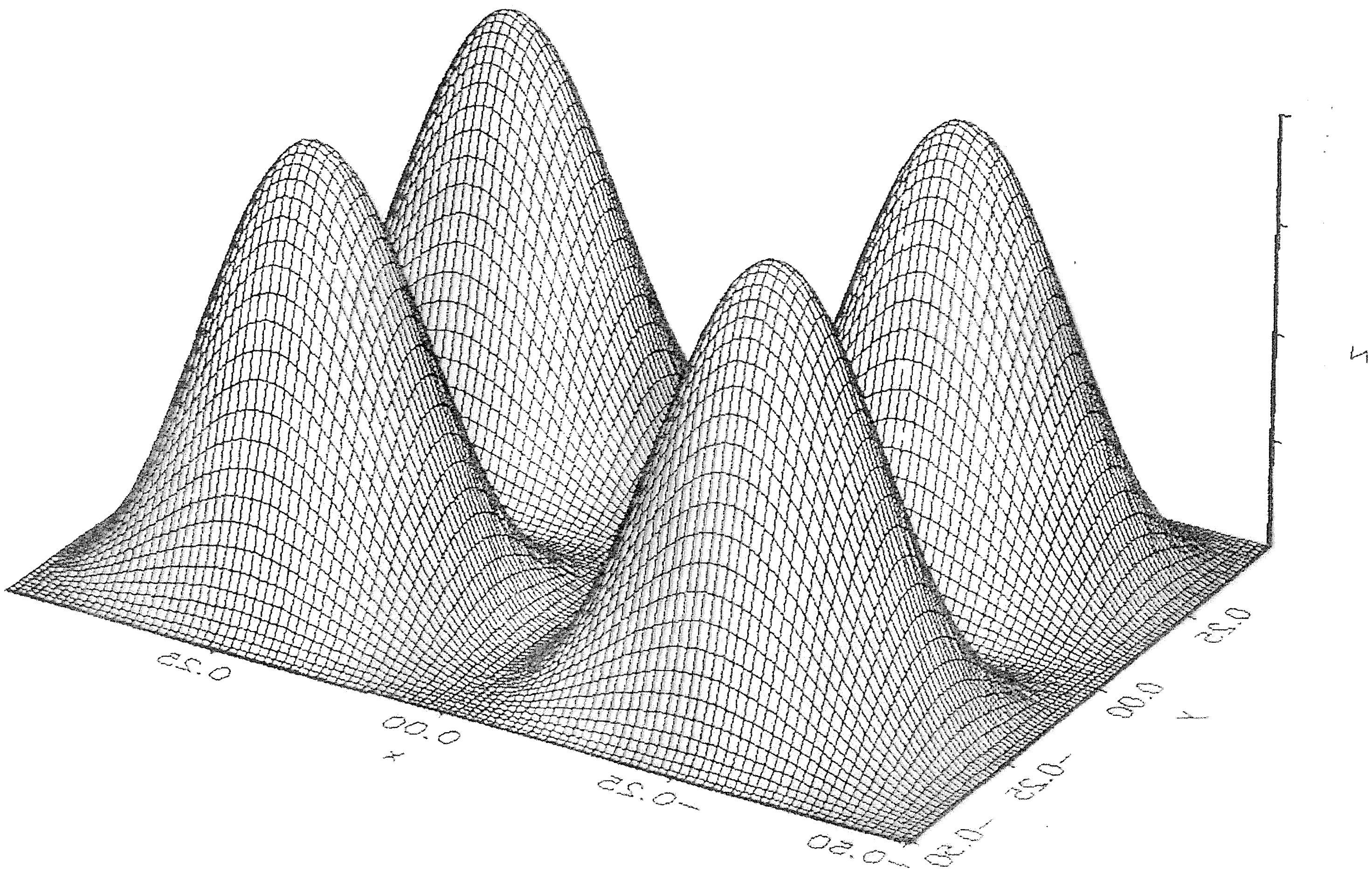
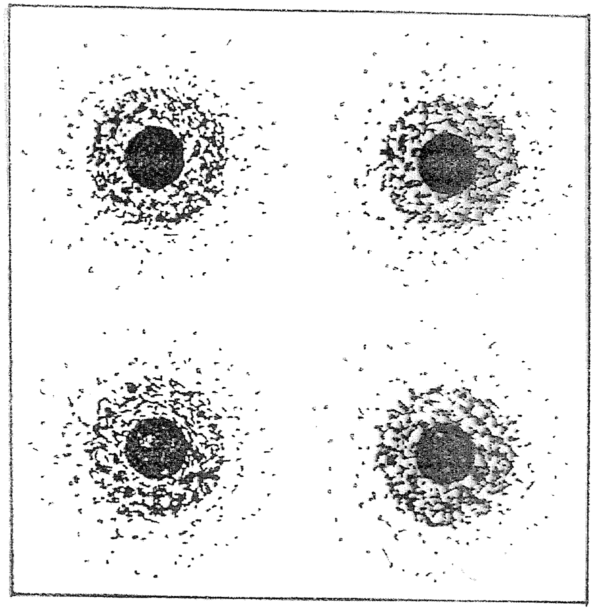
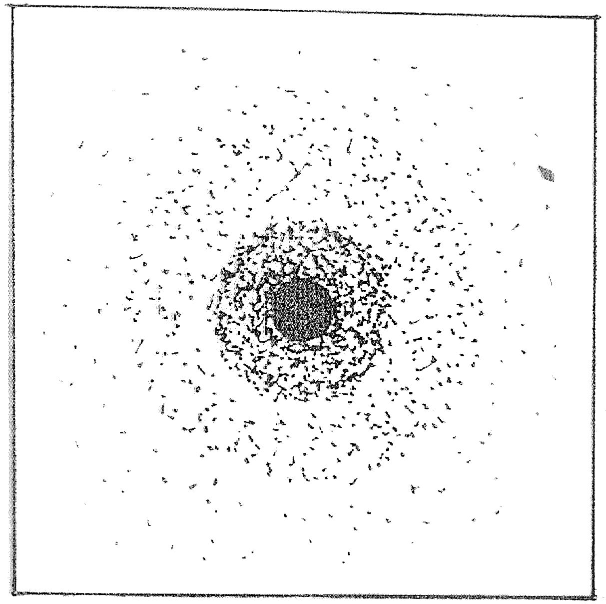
-0.50
y
-0.50
-0.25
0.00
0.25
0.50
x

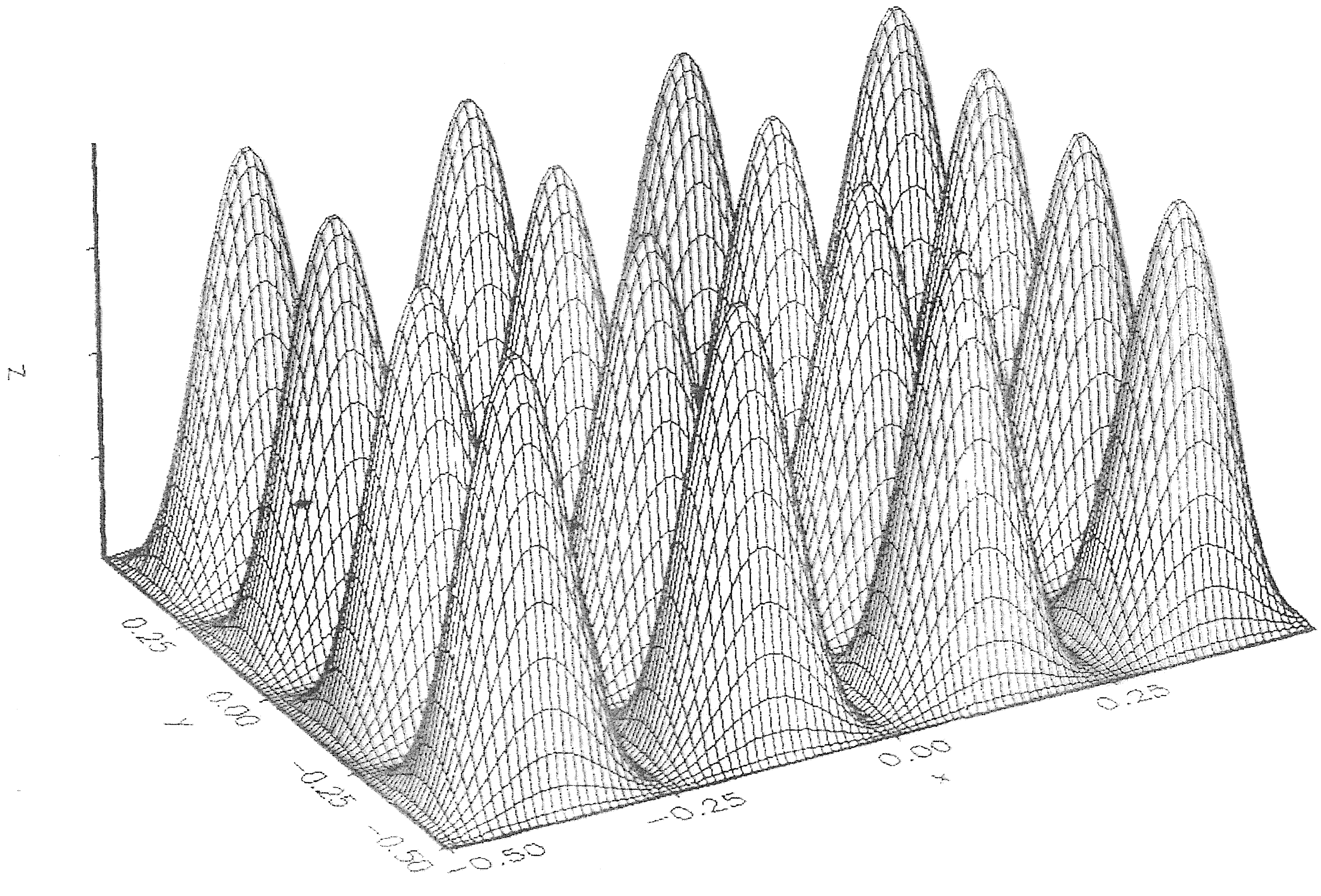

0.25
0.00
x
−0.25
−0.50
−0.50
−0.25
0.00
y
0.25
z



0.25
0.00
x
−0.25
−0.50
−0.50
−0.25
0.00
y
0.25
z







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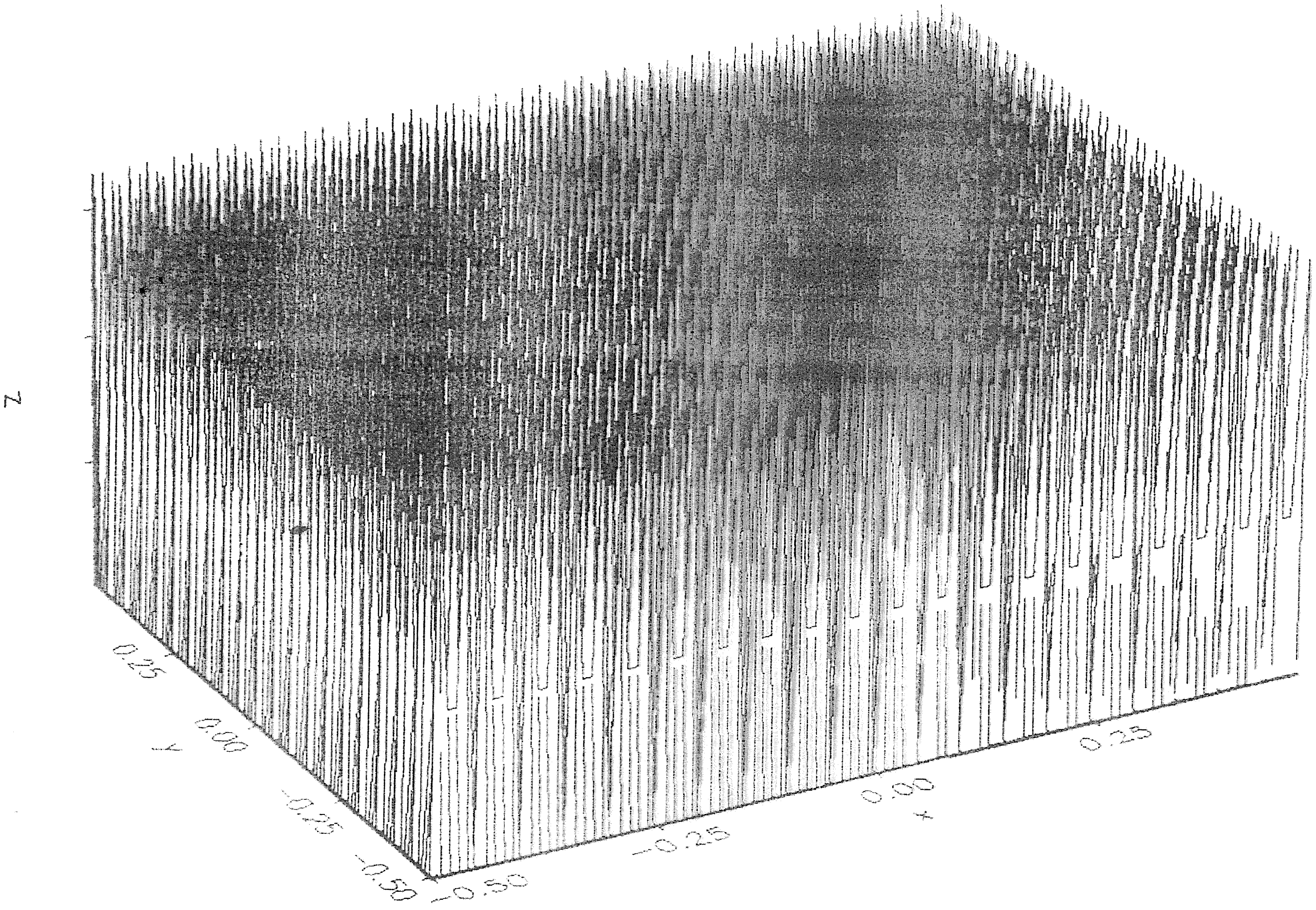

Z
0.25
0.00
Y
−0.25
−0.50
−0.50
−0.25
0.00
X
0.25

LECTURE II : RELATIVISTIC QUANTUM MECHANICS

Recipes for transition from Classical Mechanics to Quantum Mechanics:

$$H_{NR} = \frac{\vec{p}^{\,2}}{2m} + V(r) \Longrightarrow \hat{H}_{QM} = \frac{\hat{\vec{p}}^{\,2}}{2m} + \hat{V}(r) \quad (\text{II}.1)$$

where $\hat{O}$ stands for a QM operator, e.g.

$$E \to i\hbar \frac{\partial}{\partial t} \quad ; \quad \hat{\vec{p}} \to -i\hbar \vec{\nabla} \quad (\text{II}.2)$$

$$\hat{H}_{QM} \Psi(\vec{r},t) = E\, \Psi(\vec{r},t) \quad (\text{II}.3)$$

$$\therefore (\text{II}.4) \quad -\frac{\hbar^2}{2m} \nabla^2 \Psi(\vec{x},t) + V(\vec{x},t)\Psi(\vec{x},t) = i\hbar \frac{\partial \Psi}{\partial t}$$

To describe an e^- need to introduce spin by hand, i.e.

$$\vec{p} \to \left(\vec{p} - \frac{e}{c}\vec{A}\right) \quad (\text{II}.5)$$

$$\psi \to \varphi = \psi \underbrace{\begin{pmatrix} \chi_u \\ \chi_d \end{pmatrix}}_{\text{spinor}} \quad (\text{II}.6)$$

leads to the Pauli eqn.

$$(\text{II}.7) \quad i\hbar \frac{\partial \varphi}{\partial t} = \left[\frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + e\Phi - \frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B}\right]\varphi$$

$\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices.

Relativistic QM:

$$E_R^2 = \vec{p}^{\,2} c^2 + m_0^2 c^4 \xrightarrow[c=1]{} \vec{p}^{\,2} + m_0^2$$

$$E_R = \pm\sqrt{\vec{p}^{\,2} + m_0^2}$$

$\uparrow$ PROBLEM $\Rightarrow$ what is $(-)|E_R|$?

In addition
$$E_R = \pm\sqrt{(-i\hbar\vec{\nabla})^2 + m_0^2} \quad ???$$

The square-root is not "analytic".–
"Solution": Apply it twice, i.e.

$$\left[(-i\hbar\vec{\nabla})^2 c^2 + m_0^2 c^4\right]\phi(x) = \left(-i\hbar\frac{\partial}{\partial t}\right)^2\phi(x)$$

or
$$\underbrace{\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)}_{\Box^2 \equiv \partial^\mu\partial_\mu}\phi(x) + \frac{m_0^2 c^2}{\hbar^2}\phi(x) = 0$$

Klein-Gordon equation
$$\boxed{\left[\Box^2 + m_0^2\left(\frac{c^2}{\hbar^2}\right)\right]\phi(x) = 0} \quad (\text{II}.8)$$

$\Box^2$ & m_0 are Lorentz scalars, hence $\phi(x)$ is also a Lorentz scalar.

Recall solutions to $\Box^2\psi(\vec{x},t) = 0$ are plane waves. Hence try
$$\phi(x) = f(\vec{x})\, e^{-i\omega t}$$
where $f(\vec{x}) \propto e^{\pm i\vec{p}\cdot\vec{x}}$. Substituting in (II.8)

$$(-\omega^2 - \nabla^2 + m_0^2) f(\vec{x}) = (-\omega^2 + \vec{p}^{\,2} + m_0^2) f(\vec{x}) = 0$$

$$\therefore \quad \omega = \pm\sqrt{\vec{p}^{\,2} + m_0^2} = \pm E$$

the energy. The negative sign remains.

Free particle solutions $\quad \phi^{(\pm)}(x) = f(\vec{x})\, e^{\mp iEt}$

Klein Gordon current:

$$\phi^*(x)\,[\partial^\mu\partial_\mu + m_0^2]\,\phi(x) = 0$$
$$-\quad \phi(x)\,[\partial^\mu\partial_\mu + m_0^2]\,\phi^*(x) = 0$$

subtracting $\quad J_\mu = i\,(\phi^*\partial_\mu\phi - \phi\,\partial_\mu\phi^*)$

$a\overleftrightarrow{\partial}b \equiv a\partial b - (\partial a)b \qquad \boxed{J_\mu = i\,\phi^*\,\overleftrightarrow{\partial_\mu}\,\phi} \quad$ (II.9)

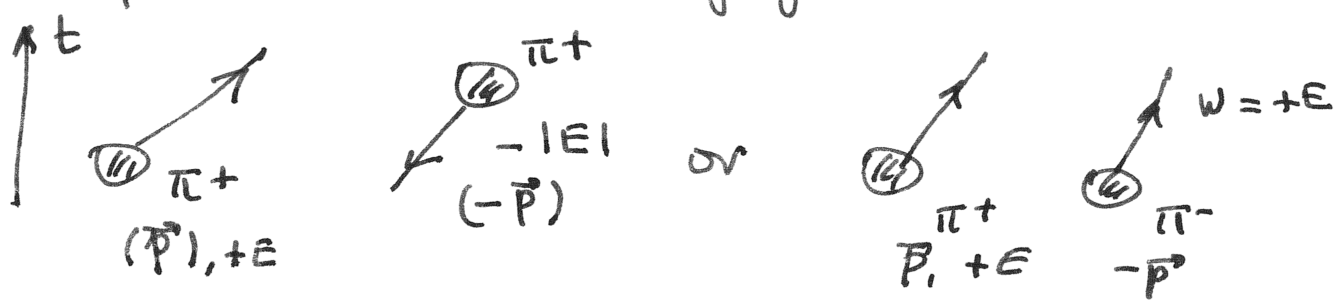
Check that $\qquad \boxed{\partial^\mu J_\mu = 0} \quad$ (II.10)

Current conservation.

$J^0 \equiv \rho\ ;\quad \vec{J} = 2\vec{p}\,|f(\vec{x})|^2;\quad$ but $\rho = \pm 2E|f|^2$

Hence, due to the $\pm$ sign, ρ is not positive definite, hence, it cannot be a probability density. Klein-Gordon: 1926 – Pauli-Weisskopf solution only in 1934: $\phi^{(-)}(x) \propto e^{i\vec{p}\cdot\vec{x}}\,e^{-iE(-t)}$!

negative E solution is a particle moving backwards in time or equivalently, an antiparticle (E<0) moving forward in time.



Free-particle solutions
$$\boxed{\phi^{(\pm)}_{\pm\vec{p}}(x) = C_{\vec{p}}\, e^{\pm i \vec{p}\cdot\vec{x}}\, e^{\mp i E t}} \qquad (\text{II}.11)$$

with $C_{\vec{p}}$ a constant.

TUTORIAL.

1) Show that $i\int d^3x\, \phi^{(+)*}_{\vec{p}\,'}(x) \overleftrightarrow{\partial_t} \phi^{(-)}_{\vec{p}}(x) = 0$

2) Show that $i\int d^3x\, \phi^{(\pm)*}_{\vec{p}\,'}(x) \overleftrightarrow{\partial_t} \phi^{(\pm)}_{\vec{p}}(x) = \pm\, \delta^{(3)}(\vec{p}-\vec{p}\,')$

if the constant is chosen as
$$|C_{\vec{p}}|^2 = \frac{1}{(2\pi)^3\, 2E} \qquad (\text{II}.12)$$

— x —

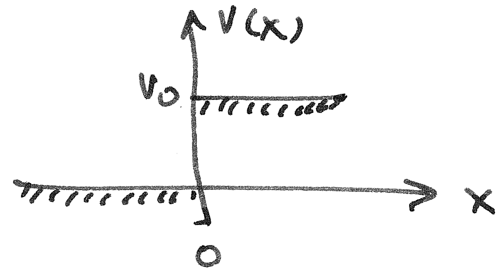
the most general KG free particle wave function is a linear superposition of (II.11), i.e.
$$\phi(x) = \int d^3p \left[a_{\vec{p}}\, \phi^{(+)}_{\vec{p}}(x) + b_{\vec{p}}\, \phi^{(-)}_{-\vec{p}}(x) \right] \qquad (\text{II}.13)$$

where $a_{\vec{p}}$ & $b_{\vec{p}}$ are Fourier coefficients, and often $\int d^3p$ is written as $\sum_{\vec{p}}$

TUTORIAL

Solve the KG equation for a step-function potential $V(x) = \begin{cases} 0 & x<0 \\ V_0 & x \geq 0 \end{cases}$ (in one dimension)

Consider $x<0$ and $x>0$, as well as $E>V_0$ and $E<V_0$. Using the boundary conditions at $x=0$ fix all coefficients in terms of e.g. that of the incoming plane wave.



$$\phi_{LEFT} = \underbrace{A e^{i\vec{p}\cdot\vec{x}}}_{\text{incident}} + \underbrace{B e^{-i\vec{p}\cdot\vec{x}}}_{\text{reflected}}$$

$$\phi_{RIGHT} = \underbrace{C e^{-i\vec{p}'\cdot\vec{x}}}_{\text{transmitted}} \qquad \begin{cases} p = \sqrt{E^2 - m_0^2} \\ p' = \sqrt{(E-V_0)^2 - m_0^2} \end{cases}$$

$$\left.\begin{aligned} A+B &= C \\ p(A-B) &= p'C \end{aligned}\right\} \quad B/A = \frac{1-p'/p}{1+p'/p} \equiv \frac{1-r}{1+r}$$

$$C/A = \frac{2}{1+p'/p} \equiv \frac{2}{1+r}$$

$$\vec{J} = -i\,\phi^* \overleftrightarrow{\partial_x} \phi \Big|_{x=0} \Rightarrow \begin{aligned} J_{inc} &= 2p|A|^2 \\ J_{refl.} &= -2p|B|^2 \\ J_{trans} &= 2p'|C|^2 \end{aligned}$$

$$\left\{ \begin{aligned} \left|\frac{J_{refl}}{J_{inc}}\right| &= \frac{|B|^2}{|A|^2} = \left|\frac{1-r}{1+r}\right|^2 \\ \left|\frac{J_{trans}}{J_{inc}}\right| &= \frac{p'}{p}\left|\frac{2}{1+r}\right|^2 \end{aligned} \right.$$

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a) $(E-V_0) < m_0$; $p' = i|p'|$, $r = i|r|$

$$\left|\frac{j_{refl}}{j_{inc}}\right| = \left|\frac{1-i|r|}{1+i|r|}\right|^2 = \frac{1-i|r|}{1+i|r|}\,\frac{1+i|r|}{1-i|r|} = 1$$

$$\left|\frac{j_{trans}}{j_{inc}}\right| = i|r|\left|\frac{2}{1+r}\right|^2 \therefore \underline{Re\left|\frac{j_{trans}}{j_{inc}}\right| = 0}$$

There is no transmission!

b) $(E-V_0) > m_0$; p' is now real!

$$\underline{r \leq 1}$$

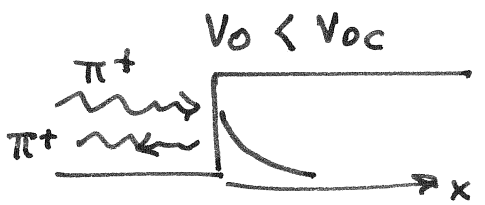
$$\left|\frac{j_{refl}}{j_{inc}}\right| < 1 , \quad \left|\frac{j_{trans}}{j_{inc}}\right| = \frac{4r}{(1+r)^2} < 1 \text{ but real!}$$

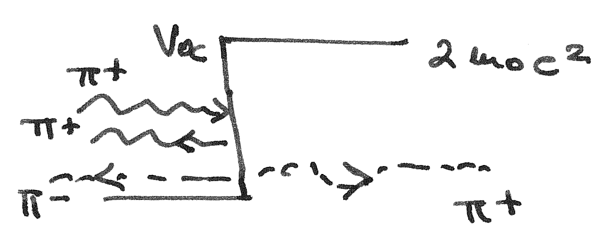
The critical V_0 is

$$(E-V_{0c})^2 = (m_0c^2)^2 \quad (E < V_{0c})$$

The minimal E is $E_{min} = m_0c^2 > 0$ (for $\vec{p}=0$)

Hence $V_{0c} = E_{min} + m_0c^2 = 2m_0c^2$





For $V_{0c} \gtrsim 2m_0c^2$ a $\pi^+\pi^-$ is produced at $x=0$. When V is $4m_0c^2$ a four-π state is produced, etc.

Dirac Equation (1934)

The Klein-Gordon eqn. can be obtained by operating with $E_R^2 = \vec{p}^2 + m_0^2 \Rightarrow (-i\partial)^2 + m_0^2$, instead of $E_R = \pm\sqrt{\vec{p}^2 + m_0^2}$.

Dirac eqn. is the result of a clever way of interpreting the square root, viz.

$$\sqrt{a^2+b^2} \neq a + b, \text{ however}$$

$$\sqrt{a^2+b^2} \overset{?}{\equiv} \alpha a + \beta b$$ if α, β are not ordinary numbers.

In fact, there is such an algebra, the Clifford algebra, viz.

$$\text{“}\sqrt{(-\partial^\mu\partial_\mu - m_0^2)}\text{”} \Longrightarrow i\gamma^\mu\partial_\mu - m_0\mathbb{1}, \quad (\text{II}.14)$$

where m_0 is assumed real, i.e. $m_0^* = m_0$, and γ^μ is a 4-vector in Minkowski space, but with elements that are not ordinary numbers,

$$\gamma^\mu = (\gamma_0, \vec{\gamma}) \; ; \quad \gamma_\mu = g_{\mu\nu}\gamma^\nu = (\gamma_0, -\vec{\gamma})$$

The properties of the components, γ_0, γ_i still need to be determined.

With (II.14) we find the Dirac eqn.

$$\boxed{(i\gamma^\mu\partial_\mu - m_0)\psi(x) = 0} \quad (\text{II}.15)$$

The Dirac equation should lead to the relativistic expression

$$\underbrace{p^\mu p_\mu = m_0^2}_{} \qquad (\text{II}.16)$$

Applying the operator $(i\gamma^\mu \partial_\mu + m_0)$ to Eq. (II.15)

$$(i\gamma^\mu \partial_\mu + m_0)(i\gamma^\nu \partial_\nu - m_0)\,\psi(x) = 0 \qquad (\text{II}.17)$$

the object $\gamma^\mu \gamma^\nu$ is a 4-tensor, which can be decomposed into symmetric & antisymmetric pieces, i.e.

$$\gamma^\mu\gamma^\nu = \frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) + \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) \qquad (\text{II}.18)$$

Recalling that the product of a symmetric & an antisymmetric tensor vanishes (prove it by yourself), and $\partial_\mu\partial_\nu$ is a symmetric tensor, one finds from (II.17)

$$\left[-\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\,\partial_\mu\partial_\nu - m_0^2\right]\psi(x) = 0, \qquad (\text{II}.19)$$

where the anti-commutator is defined as

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \qquad (\text{II}.20)$$

Using $-\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu = \frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\,p_\mu\, p_\nu$ (II.21)

Eq. (II.19) becomes

$$\left[\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\,p_\mu p_\nu - m_0^2\right]\psi(x) = 0 \qquad (\text{II}.22)$$

In order to recover Eq. (II.16) the γ's must satisfy

$$\boxed{\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu}} \quad (\text{II}.23)$$

which defines the (anti-commuting) Clifford algebra. We obtain now some other properties of γ^μ:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \Rightarrow \begin{cases} \gamma^0\gamma^0 + \gamma^0\gamma^0 = 2(\gamma^0)^2 = 2g^{00} = 2 \\ \gamma^i\gamma^i + \gamma^i\gamma^i = 2(\gamma^i)^2 = 2g^{ii} = -2 \end{cases}$$

$$\boxed{(\gamma^0)^2 = +1} \quad \boxed{(\gamma_i)^2 = -1} \quad (\text{II}.24)$$

Eigenvalue equations:

$(\gamma_0)^2 = 1\,; \quad \gamma_0(\gamma_0^2) \equiv (\gamma_0)^3 = \gamma^0, \quad \therefore$

$$\boxed{[(\gamma^0)^2 - 1]\,\gamma^0 = 0} \quad (\text{II}.25)$$

Hence γ^0 has eigenvalues ± 1

$(\gamma_i)^2 = -1\,; \quad \gamma_i(\gamma_i)^2 = -\gamma_i\,, \quad \therefore$

$$\boxed{[(\gamma_i)^2 + 1]\,\gamma_i = 0} \quad (\text{II}.26)$$

Hence γ^i has eigenvalues $\pm i$.

From $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, with $\mu = 0$, $\nu = i$ we find

$\gamma^0\gamma^i + \gamma^i\gamma^0 = 0$ multiplying by γ^0 gives

$(\gamma^0)^2\gamma^i + \gamma^0\gamma^i\gamma^0 = 0$ or

$$\boxed{\gamma^i = -\gamma^0\gamma^i\gamma^0} \quad (\text{II}.27)$$

The γ's cannot be ordinary (c-) numbers, i.e. from (II.23) $\gamma^i\gamma^0 + \gamma^0\gamma^i = 0 \therefore \gamma^i\gamma^0 = -\gamma^0\gamma^i$

In addition to (II.27) we find an equation for γ^0, using $\gamma^0\gamma^i + \gamma^i\gamma^0 = 0$ & multiplying by γ^i and using $(\gamma_i)^2 = -1$:

$$\boxed{\gamma^0 = \gamma^i\gamma^0\gamma^i} \quad (\text{II}.28)$$

Computing the trace of γ's:

From (II.28): $\mathrm{Tr}(\gamma^0) = \mathrm{Tr}(\gamma^i\gamma^0\gamma^i) = \mathrm{Tr}(\gamma^0\gamma^{i^2})$
$= -\mathrm{Tr}(\gamma^0)$

$$\therefore \quad \boxed{\mathrm{Tr}(\gamma^0) = 0} \quad (\text{II}.29)$$

Similarly, from II.27) $\mathrm{Tr}(\gamma^i) = -\mathrm{Tr}(\gamma^0\gamma^i\gamma^0) = -\mathrm{Tr}(\gamma^i)$

$$\boxed{\mathrm{Tr}(\gamma^i) = 0} \quad (\text{II}.30)$$

Hence:

$$\underline{\gamma_0 \,\&\, \gamma_i}\left\{\begin{array}{l}\text{eigenvalues } \pm 1 \,\&\, \pm i, \text{ respectively} \\ \text{zero trace [trace} = \textstyle\sum \text{eigenvalues]} \\ \text{anti-commuting}\end{array}\right.$$

Can be REPRESENTED by MATRICES of even dimension, $N \geq 2$.

If $N = 2 \Rightarrow$ Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ & $\mathbb{1}$

but $\mathbb{1}$ is NOT anticommuting with σ_i.

Next, $\underline{\underline{N = 4}}$. OK. 16 matrices ($N^2 = 16$)

so that $\psi(x)$, Dirac wave function, is a 4-component object.

The 16 matrices:

$$\Gamma^S \equiv \mathbb{1} \; ; \; \Gamma^V_\mu \equiv \gamma_\mu \; ; \; \Gamma^T_{\mu\nu} \equiv \sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$$

$$\Gamma^P \equiv \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5 \; ; \; \Gamma^A_\mu \equiv \gamma_5\gamma_\mu = -\gamma_\mu\gamma_5$$

$$\Gamma^S \Rightarrow 1 \; ; \; \Gamma^V_\mu \Rightarrow 4 \; ; \; \Gamma^A_\mu \Rightarrow 4 \; ; \; \Gamma^P \Rightarrow 1 \; ; \; \Gamma^T_{\mu\nu} \Rightarrow 6$$

TOTAL # = 16

Recall: $\begin{cases} \sigma_{ij} = i\gamma_i\gamma_j \; (i \neq j) \\ \sigma_{00} = \sigma_{ii} = 0 \end{cases}$

"Bjorken - Drell" convention:

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} ; \; \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} ; \; \gamma_5 = \gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \; \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \; \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The γ_5 convention is not "universal", hence PAY ATTENTION!

Dirac eqn: 4 - solutions: $\begin{cases} E>0 \text{ spin } \frac{1}{2} \; (\uparrow) \text{ or } (\downarrow) \\ E<0 \text{ spin } \frac{1}{2} \; (\uparrow) \text{ or } (\downarrow) \end{cases}$

<u>and</u> μ (magnetic moment) $= \mu_B = \dfrac{e\hbar}{2m}$

$$\mu_{e^-}\Big|_{EXP} = \mu_B(1+a) \quad \text{with } a \neq 0, \; a_{EXP} \cong 0.0011$$

g - factor $\quad g \Rightarrow \quad a \equiv \dfrac{g-2}{2}$

Hence a is the deviation of g from 2.

$a \simeq 0.1\%$!

Finding a Dirac Hamiltonian, analogous to the non-relativistic Schrödinger Hamiltonian

$$i\hbar \frac{\partial \Psi(\vec{x},t)}{\partial t} = \hat{H}_S \Psi(\vec{x},t) \qquad (\text{II}.31)$$

Start from $(i\gamma^\mu \partial_\mu - m_0)\psi(x) = 0 \Rightarrow i\gamma^0 \frac{\partial}{\partial t}\psi + i\vec{\gamma}\cdot\vec{\nabla}\psi = m_0\psi$

& multiply by γ^0 to obtain

$$i\frac{\partial \psi}{\partial t} = (-i\gamma_0\vec{\gamma}\cdot\vec{\nabla} + m_0\gamma_0)\psi(x) \equiv \hat{H}_D\psi(x) \qquad (\text{II}.32)$$

with

$$\boxed{\begin{aligned}\hat{H}_D &\equiv -i\gamma_0\vec{\gamma}\cdot\vec{\nabla} + m_0\gamma_0 \\ &= \gamma_0\vec{\gamma}\cdot\vec{p} + m_0\gamma_0\end{aligned}} \qquad (\text{II}.33)$$

since $\hat{\vec{p}}$ is an Hermitian operator, and so must be $\hat{H}$, we find new relations for the γ_μ's, viz.

$$\hat{H}^\dagger = (\gamma_0\gamma_i)^\dagger \hat{p}_i + m_0\gamma_0^\dagger = \gamma_0\gamma_i p_i + m_0\gamma_0$$

$$\Rightarrow \boxed{\begin{aligned}\gamma_0^\dagger &= \gamma_0 \\ \gamma_i^\dagger &= -\gamma_i\end{aligned}} \qquad (\text{II}.34)$$

or in one equation:

$$\boxed{\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0} \qquad (\text{II}.35)$$

NOTATION

$$\boxed{\gamma^\mu p_\mu \equiv \not{p} \; ; \quad \gamma^\mu a_\mu \equiv \not{a}} \qquad (\text{II}.36)$$

<u>TUTORIAL</u>

Obtain the Dirac equation for the "Hermitian Adjoint" spinor, i.e. $\bar{\Psi}(x) \equiv \psi^{\dagger}(x)\,\gamma_0$.

Hint: Take the Hermitian (dagger $\dagger$) equation & use the properties of the γ's to find

$$\boxed{\bar{\Psi}(x)\,(i\overleftarrow{\not{\partial}} + m_0) = 0} \qquad (\text{II}.37)$$

where the $\overleftarrow{\not{\partial}}$ operator acts on the left.

<u>TUTORIAL</u>

Find the (conserved) Dirac current.

Hint: Start with the Dirac eqn. & multiply it by $\bar{\Psi}(x)$ on the left, w/ the (II.37) multiplied by $\Psi(x)$ on the right, and subtract the latter from the former to find

$$\boxed{i\,\partial_\mu\left(\bar{\Psi}(x)\,\gamma^\mu\,\Psi(x)\right) = 0} \qquad (\text{II}.38)$$

where the current $\bar{\Psi}\gamma^\mu\Psi$ is identified with the electromagnetic current

$$\boxed{J^\mu(x) \equiv \bar{\Psi}(x)\,\gamma^\mu\,\Psi(x)} \qquad (\text{II}.39)$$

with $|e| = +1$. –

<u>TUTORIAL</u> :

Show that $J^\mu(x)$ is Hermitian.

Dirac equation in p-space:

$$(i \not\partial - m_0)_{ab} \, \psi_b(x) = 0 \qquad (a,b = 1,2,3,4)$$

free-particle: $\psi_a(x) = w_a(p)\, e^{-ip\cdot x}$

substituting in the eqn. gives

$$\boxed{(\not p - m_0)\, w(p) = 0} \qquad (\text{II}.40)$$

which is Dirac eqn. in p-space.

TUTORIAL:

Show that for any 4-vectors a^μ & b^μ

$$\boxed{\not a \not b = a\cdot b - i\, \sigma_{\mu\nu}\, a^\mu b^\nu} \qquad (\text{II}.41)$$

Hint: use Eq. (II.23) together with $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$

TUTORIAL:

1) Show that $\not a \not a = a^2$

2) Show that (II.40) leads to the relativistic energy $\pm E = p_0 = \pm\sqrt{\vec p^{\,2} + m_0^{\,2}}$.

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The solutions to the free particle eqn. in terms of $w_a(\vec p)$ can be rewritten in terms of two spinors $u(\vec p)$ & $v(\vec p)$ representing positive energy & spin-up/down, and negative energy & spin-up/down, i.e.

$$\psi^{(\pm)}(x) = \begin{cases} u(\vec{p})\, e^{-ip\cdot x} & (E>0) \\ v(\vec{p})\, e^{ip\cdot x} & (E<0) \end{cases} \qquad (\text{II}.42)$$

TUTORIAL

substituting Eq. (II.42) into the Dirac show that the spinors $u(\vec{p})$ & $v(\vec{p})$ satisfy

$$\begin{cases} (\not{p} - m_0)\, u(\vec{p}) = 0 \\ (\not{p} + m_0)\, v(\vec{p}) = 0 \end{cases} \qquad (\text{II}.43)$$

$$\begin{cases} \bar{u}(\vec{p})\, (\not{p} - m_0) = 0 \\ \bar{v}(\vec{p})\, (\not{p} + m_0) = 0 \end{cases} \qquad (\text{II}.44)$$

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EXPLICIT SOLUTIONS

$$u(\vec{p},s) = N \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m_0}\chi_s \end{pmatrix} \; ; \; v(\vec{p},s) = N \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m_0}\eta_s \\ \eta_s \end{pmatrix} \qquad (\text{II}.45)$$

$(s = 1, 2)$

where $\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $\chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; $\eta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $\eta_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

& the normalization factor $N = \sqrt{E+m_0}$ or the Bjorken-Drell value $N_{BD} = \frac{\sqrt{E+m_0}}{\sqrt{2m_0}}$.

TUTORIAL

Show that $(\vec{\sigma}\cdot\vec{p})^2 = \vec{p}^{\,2}\, \mathbb{1}$ with $\mathbb{1}$ the 2x2 unit matrix. HINT: use $\sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k$ $(i \neq j)$ & $\sigma_i\sigma_i = 1$.

The explicit free particle solutions are

$$u(\vec{p}, s=1) = N \begin{pmatrix} 1 \\ 0 \\ p_3/(E+m) \\ p_+/(E+m) \end{pmatrix} ; \quad u(p, s=2) = N \begin{pmatrix} 0 \\ 1 \\ p_-/(E+m) \\ -p_3/(E+m) \end{pmatrix}$$

$$v(\vec{p}, s=1) = N \begin{pmatrix} p_3/(E+m) \\ p_+(E+m) \\ 1 \\ 0 \end{pmatrix} ; \quad v(p, s=2) = N \begin{pmatrix} p_-/(E+m) \\ -p_3/(E+m) \\ 0 \\ 1 \end{pmatrix}$$

$$\boxed{p_\pm \equiv p_1 \pm i p_2}$$

TUTORIAL

Show that

1) $u^\dagger(\vec{p}, s)\, u(\vec{p}, s') = \# \, \delta_{ss'}$; $\# = \begin{cases} 2E \\ E/m \text{ (Bjorken-Drell)} \end{cases}$

2) $v^\dagger(\vec{p}, s)\, v(\vec{p}, s') = \# \, \delta_{ss'}$

3) show that $u(\vec{p}, s)$ is orthogonal to $v(-\vec{p}, s')$ but not to $v(\vec{p}, s)$.

4) $\hat{H}_D \, u(\vec{p}, s) = E \, u(\vec{p}, s)$

5) $\hat{H}_D \, v(-\vec{p}, s) = -E \, v(-\vec{p}, s)$

6) $\bar{u}(\vec{p}, s)\, u(\vec{p}, s') = \begin{pmatrix} 2m \\ \text{or} \\ 1 \end{pmatrix} \delta_{ss'}$

top value is our normalization bottom is Bj & Drell

7) $\bar{v}(\vec{p}, s)\, v(\vec{p}, s') = \begin{pmatrix} -2m \\ \text{or} \\ -1 \end{pmatrix} \delta_{ss'}$

8) $\bar{u}(\vec{p}, s)\, v(\vec{p}, s') = \bar{v}(\vec{p}, s)\, u(\vec{p}, s') = 0$

9) $\sum_{s=1}^{2} \left[u_a(\vec{p}, s)\, \bar{u}(\vec{p}, s) - v(\vec{p}, s)\, \bar{v}(\vec{p}, s) \right] = \begin{pmatrix} 2m \\ \text{or} \\ 1 \end{pmatrix}$

LECTURE 3: RELATIVISTIC QUANTUM FIELDS

Classical field $\Longrightarrow$ Quantum Field Operator in Fock (particle) space.

<u>RQM</u>: Klein Gordon: $(\partial^\mu \partial_\mu + m^2)\phi(x) = 0.$ (III.1)

$\phi(x) \Longrightarrow \hat{\phi}(x)$. The KG equation <u>and</u> its wave-function are treated as "classical" objects becoming a "classical" field which is then quantized by transforming the Fourier coefficients as quantum operators in Fock space.

$$\phi_{KG}(x) = \int d^3p \left[a_{\vec{p}}\, \phi^{(+)}_{\vec{p}}(x) + b^*_{\vec{p}}\, \phi^{(-)}_{-\vec{p}}(x) \right] \quad (\text{III}.2)$$

(with $\phi^\dagger_{KG} \neq \phi_{KG}$). Now $\phi_{KG} \Longrightarrow \hat{\phi}_{KG}$ with

$$\hat{\phi}_{KG}(x) = \int d^3p \left[\hat{a}_{\vec{p}}\, \phi^{(+)}_{\vec{p}}(x) + \hat{b}^\dagger_{\vec{p}}\, \phi^{(-)}_{-\vec{p}}(x) \right] \quad (\text{III}.3)$$

$$\phi^{(\pm)}_{\pm}(x) = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_{\vec{p}}}}\, e^{\pm i \vec{p}\cdot\vec{x}}\, e^{\mp iEt} \quad (\text{III}.4)$$

The operators $\hat{a}_{\vec{p}}$ & $\hat{b}_{\vec{p}}$ will eventually create / annihilate particles & anti-particles by acting on the vacuum state $|0\rangle$.

The KG quantum field Lagrangian is

$$\boxed{\hat{\mathcal{L}}_{KG} = \partial^\mu \hat{\phi}^\dagger(x)\, \partial_\mu \hat{\phi}(x) - m_0^2\, \hat{\phi}^\dagger(x)\, \hat{\phi}(x)} \quad (\text{III}.5)$$

which using (I.13) leads to $(\partial^\mu\partial_\mu + m_0^2)\begin{pmatrix}\hat{\phi}(x)\\ \hat{\phi}^\dagger(x)\end{pmatrix} = 0$.

This procedure is usually called "second quantization".

Equation (III.5) can be rewritten as

$$\hat{\mathcal{L}}_{KG} = -\frac{1}{2}\,\hat{\phi}(x)\left[\partial^\mu\partial_\mu \hat{\phi}^\dagger(x) + m_0^2\, \hat{\phi}^\dagger(x)\right] \quad (\text{III}.6)$$
$$- \frac{1}{2}\,\hat{\phi}^\dagger(x)\left[\partial^\mu\partial_\mu \hat{\phi}(x) + m_0^2\, \hat{\phi}(x)\right] + \frac{1}{2}\partial^\mu T_\mu(x)$$

with $T_\mu(x) \equiv \hat{\phi}(x)\,\partial_\mu \hat{\phi}^\dagger(x) + \hat{\phi}^\dagger(x)\,\partial_\mu \hat{\phi}(x)$ (III.7)

<u>TUTORIAL</u>: Why do the 2nd & 3rd terms above not vanish on account of the KG equation?

<u>TUTORIAL</u>:

1) Show that (III.6) is identical to (III.5)

2) Show that $\int d^4x \; \partial^\mu T_\mu(x)$ does not contribute to δS (S is the KG action)
HINT: Recall that the variation δS is performed at fixed times (t_1, t_2) and over a fixed three-dimensional surface (where the fields do not change).

3) Compute $\delta S = 0$ using (III.6) and show that the KG equations for $\hat{\phi}(x)$ & $\hat{\phi}^\dagger(x)$ are obtained.

The Hamiltonian:

$$\hat{\mathcal{H}} = \sum_i \hat{\pi}_i \, \partial_0 \hat{\phi}_i - \hat{\mathcal{L}} \qquad (\text{III}.8)$$

$$\text{with} \quad \hat{\pi}_i = \frac{\partial \hat{\mathcal{L}}}{\partial(\partial_0 \hat{\phi}_i)} \qquad (\text{III}.9)$$

TUTORIAL

Using (III.8) & (III.9) show that

$$\hat{H} \equiv \int d^3x \, \hat{\mathcal{H}}$$

$$= -\int d^3x \, \hat{\phi}^\dagger(x) \, \overleftrightarrow{\partial_0} \, (\partial_0 \hat{\phi}(x))$$

$$= \int d^3p \, E_{\vec{p}} \left(\hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{p}} + \hat{b}_{\vec{p}} b^\dagger_{\vec{p}} \right) \qquad (\text{III}.10)$$

and we return shortly to discuss the last term above (see page 50).

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Fock Space

$|0\rangle$ Vacuum $\Rightarrow$ no particles, such that $\langle 0|0\rangle = 1$

By "no particles" it is meant no "permanent" particles, i.e. it excludes vacuum fluctuations.

$$\hat{a}_{\vec{p}} |0\rangle = \hat{b}_{\vec{p}} |0\rangle = 0 \quad \text{annihilation operators}$$

$$\hat{a}^\dagger_{\vec{p}} |0\rangle \propto |\vec{p}\rangle \quad \& \quad \hat{b}^\dagger_{\vec{p}} |0\rangle \propto \overline{|\vec{p}\rangle}$$

particle & antiparticle creation ops.

$$\langle 0 | \hat{a}_{\vec{p}} \, \hat{a}^\dagger_{\vec{p}} |0\rangle = 1; \quad \langle 0 | \hat{b}_{\vec{p}} \hat{b}^\dagger_{\vec{p}} |0\rangle = 1$$

NOTICE: $(\hat{a}_{\vec{p}} |n\rangle)^\dagger = \langle n | \hat{a}_{\vec{p}}^\dagger = 0$

Commutation Relations (Algebra of ops.)

QM: $[q_i, p_j] = i(\hbar)\delta_{ij}$; $[q_i, q_j] = [p_i, p_j] = 0$

Postulate (equal time commutators)

$$[\hat{\phi}(\vec{x},t), \hat{\pi}(\vec{y},t)] = i\,\delta^{(3)}(\vec{x}-\vec{y}) \quad (\text{III}.11)$$

$$[\hat{\phi}(\vec{x},t), \hat{\phi}(\vec{y},t)] = [\hat{\pi}(\vec{x},t), \hat{\pi}(\vec{y},t)] = 0 \quad (\text{III}.12)$$

$$\hat{\pi} \equiv \partial_0 \hat{\phi}^\dagger, \quad \text{viz.} \begin{cases} \mathcal{L} = \partial^\mu \hat{\phi}^\dagger \partial_\mu \hat{\phi} - m_0^2 \hat{\phi}^\dagger \hat{\phi} \\ \dfrac{\partial \mathcal{L}}{\partial(\partial_0 \hat{\phi})} = \partial^0 \hat{\phi}^\dagger = \partial_0 \hat{\phi}^\dagger \end{cases} \quad (\text{III}.13)$$

TUTORIAL:

Show that (III.13) leads to

$$[\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{p}'}] = \delta^{(3)}(\vec{p}-\vec{p}') \quad (\text{III}.14)$$

$$[\hat{b}_{\vec{p}}, \hat{b}^\dagger_{\vec{p}'}] = \delta^{(3)}(\vec{p}-\vec{p}') \quad (\text{III}.15)$$

and all other commutators vanish identically.

HINT: Start from $[\hat{\phi}(\vec{x},t), \hat{\pi}(\vec{y},t)]$ Eq. (III.11) & substitute the Fourier expansion (III.3). Next perform the integrations. Use the relations

$$\int d^3p'\, \delta^{(3)}(\vec{p}'-\vec{p})\, e^{-i\vec{p}\cdot\vec{y}}\, e^{i\vec{p}'\cdot\vec{x}} =$$
$$= \int d^3p'\, \delta^{(3)}(\vec{p}'-\vec{p})\, e^{-i\vec{p}'\cdot\vec{y}}\, e^{i\vec{p}\cdot\vec{x}} \quad (\text{III}.16)$$

$$\delta(\vec{x}-\vec{y}) = \int \frac{d^3p}{(2\pi)^3} \Big\{ \frac{1}{2}\int d^3p'\, e^{i(\vec{p}'\cdot\vec{x} - \vec{p}\cdot\vec{y})}\, \delta^{(3)}(\vec{p}'-\vec{p}) + \frac{1}{2}\int d^3p'\, e^{i(\vec{p}\cdot\vec{x} - \vec{p}'\cdot\vec{y})}\, \delta^{(3)}(\vec{p}'-\vec{p}) \Big\} \quad (\text{III}.17)$$

The rewriting of $\hat{H}$ in Eq. (III.10).

It is needed in order that $\hat{H}|0\rangle = 0$, so one needs to shift $\hat{b}_{\vec{p}}\hat{b}^{\dagger}_{\vec{p}}$ into $\hat{b}^{\dagger}_{\vec{p}}\hat{b}_{\vec{p}}$.

From $\langle 0|\hat{b}_{\vec{p}}\hat{b}^{\dagger}_{\vec{p}}|0\rangle = 1$ (see bottom of page 48)

we can also write it as

$$\langle 0|\hat{b}_{\vec{p}}\hat{b}^{\dagger}_{\vec{p}} \pm \hat{b}^{\dagger}_{\vec{p}}\hat{b}_{\vec{p}}|0\rangle = 1 \quad (\text{III}.18)$$

as $\hat{b}_{\vec{p}}|0\rangle = 0$. This means, at the operator level that

$$\hat{b}_{\vec{p}}\hat{b}^{\dagger}_{\vec{p}} = 1 \pm \hat{b}^{\dagger}_{\vec{p}}\hat{b}_{\vec{p}} \quad (\text{III}.19)$$

The Hamiltonian, Eq. (III.10) then becomes

$$\hat{H} = \int d^3p\, E_{\vec{p}}\left(\hat{a}^{\dagger}_{\vec{p}}\hat{a}_{\vec{p}} \pm \hat{b}^{\dagger}_{\vec{p}}\hat{b}_{\vec{p}}\right) + \int d^3p\, E_{\vec{p}}$$

$$\left(\hat{H} - \int d^3p\, E_{\vec{p}}\right) \Rightarrow \hat{H} = \int d^3p\, E_{\vec{p}}\left(\hat{a}^{\dagger}_{\vec{p}}\hat{a}_{\vec{p}} + \hat{b}^{\dagger}_{\vec{p}}\hat{b}_{\vec{p}}\right) \quad (\text{III}.20)$$

where we have chosen the + sign to make $\hat{H}$ positive definite, and the divergent integral $\int d^3p\, E_{\vec{p}}$ has been absorbed in $\hat{H}$.

Eq. (III.20) then satisfies $\hat{H}|0\rangle = 0$, so by definition the vacuum has zero energy.

Normalization of 1-particle state:

$$|p\rangle = C^{-1}_{\vec{p}}\,\hat{a}^{\dagger}_{\vec{p}}|0\rangle \text{ with } C_{\vec{p}} = \frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2E_{\vec{p}}}} \quad (\text{III}.21)$$

<u>TUTORIAL</u>

Show that $\hat{H}|p\rangle = E_p|p\rangle$ & $\hat{H}\,\overline{|p\rangle} = E_p\,\overline{|p\rangle}$