# Physics Honours: Standard Model

## Tutorial Sheet 3

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## Question 1

Show the following relationships between the unitary and hermitian matrices:

a) Any $n \times n$ unitary matrix $U^\dagger U = 1$ can be written as

$$U = \exp(iH)$$

where $H$ is hermitian, $H^\dagger = H$.

b) $\det U = 1$ implies that $H$ is traceless.

**Remark**: This result means that $n \times n$ unitary matrices with unit determinant can be generated by $n \times n$ traceless hermitian matrices.

## Question 2

The $n \times n$ unitary matrices with unit determinant form the $SU(n)$ group.

a) Show that it has $n^2 - 1$ independent group parameters.

b) Show that the maximum number of mutually commuting matrices in an $SU(n)$ group is $(n-1)$. (This is the **rank** of the group.)

## Question 3

This problem illustrates the special property of the $SU(2)$ representations, their being equivalent to their complex conjugate representations.

a) For every $2 \times 2$ unitary matrix $U$ with unit determinant, show there exists a matrix $S$ which connects $U$ to its complex conjugate matrix $U^*$ through the similarity transformation

$$S^{-1} U S = U^* \ .$$

b) Suppose $\psi_1$ and $\psi_2$ are the bases for the spin-$\frac{1}{2}$ representation of $SU(2)$ having eigenvalues of $\pm\frac{1}{2}$ for the diagonal generator $T_3$;

$$T_3 \psi_1 = \frac{1}{2}\psi_1 \ \text{ and } \ T_3 \psi_2 = -\frac{1}{2}\psi_2 \ ,$$

calculate the eigenvalues of $T_3$ operating on $\psi_1^*$ and $\psi_2^*$, respectively.

## Question 4

a) Show that if $A$ and $B$ are two $n \times n$ matrices, we have the Baker-Hausdorff relation

$$e^{iA}Be^{-iA} = B + i[A,B] + \frac{i}{2!}[A,[A,B]] + \ldots + \frac{i^n}{n!}[A,[A,\ldots[A,B]\ldots]] + \ldots$$

b) Show that the matrix $B$ is invariant (up to a phase) under the transformations generated by the matrix $A$, if these two matrices satisfy the commutation relation of $[A,B] = B$.

## Question 5

Prove the identity for $2 \times 2$ unitary matrices generated by Pauli matrices $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$:

$$\exp(i\vec{r} \cdot \vec{\sigma}) = \cos r + (\hat{r} \cdot \vec{\sigma}) \sin r$$

where $r = |\vec{r}|$ is the magnitude of the vector $\vec{r}$ and $\hat{r} = \vec{r}/r$ is the unit vector.

## Question 6

Consider the non-relativistic Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\psi(\vec{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})\right]\psi(\vec{r},t)$$

Obtain the conserved probability 4-current

$$j^\mu \equiv (c\rho, \vec{j})$$

$$\rho = \psi^*\psi \qquad ; \qquad \vec{j} = \frac{\hbar}{2im}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)$$

# Answers

## Question 1

a) The matrix $U$ can always be diagonalised by some unitary matrix $V$

$$VUV^\dagger = U_d$$

where $U_d$ is a diagonal matrix satisfying the unitarity condition $U_d U_d^\dagger = 1$. This implies that each of the diagonal elements can be expressed as a complex number with unit magnitude $e^{i\alpha}$.

$$U_d = \begin{pmatrix} e^{i\alpha_1} & & & \\ & e^{i\alpha_2} & & \\ & & \ddots & \\ & & & e^{i\alpha_n} \end{pmatrix}$$

where $\alpha_i$'s are real. It is then straightforward to see the equality $U_d = e^{iH_d}$, where $H_d$ is a real diagonal matrix: $H_d = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n)$. We then have

$$U = V^\dagger U_d V = V^\dagger e^{iH_d} V = e^{iH}$$

with $H = V^\dagger H_d V$. Because $H_d$ is real and diagonal, the matrix $H$ is hermitian:

$$H^\dagger = \left(V^\dagger H_d V\right)^\dagger = V^\dagger H_d^\dagger V = H \ .$$

b) From the matrix identity $e^{\mathrm{Tr}A} = \det(e^A)$, we have for $U = e^{iH}$

$$e^{i\mathrm{Tr}H} = \det(e^{iH}) = \det U \ .$$

Thus $\det U = 1$ implies that $\mathrm{Tr}H = 0$.

## Question 2

a) To count the number of independent group parameters, it is easier to do so through the generator matrix. From the previous problem, we have $U = e^{iH}$, where $H$ is an $n \times n$ traceless hermitian matrix. For a general hermitian matrix, the diagonal elements must be real, $H_{ii} = H_{ii}^*$. Because of the traceless condition, this corresponds to $(n-1)$ independent parameters. There are altogether $(n^2 - n)$ off-diagonal elements and thus $(n^2 - n)$ independent parameters because each complex element corresponds to two real parameters, yet this factor of two is cancelled by the hermitian conditions $H_{ij} = H_{ji}^*$. Consequently, we have a total of $(n - 1 + n^2 - n) = (n^2 - 1)$ independent parameters.

b) From the discussion in part a) we already know that there are $n-1$ independent diagonal $SU(n)$ matrices, which obviously must be mutually commutative. On the other hand, if there were more than $n-1$ mutually commuting matrices, they could all be diagonalised simultaneously, thus yielding more than $n-1$ independent diagonal matrices. This is impossible for $n \times n$ traceless hermitian generating matrices.

## Question 3

a) We will prove this by explicit construction. Question 1 taught us that the unitary matrix $U$ can be expressed in terms of its generating matrix $U = \exp iH$. Thus the matrix $S$, if it exists, must have the property of

$$S^{-1}HS = -H^*$$

so that $S^{-1}US = S^{-1}(\exp iH)S = U^* = \exp(-iH^*)$. The generating matrix $H$, being a $2 \times 2$ traceless hermitian matrix, can be expanded in terms of the Pauli matrices

$$H = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3$$

with real coefficients of expansion $a_i$. Since $\sigma_1$ and $\sigma_3$ are real, $\sigma_2$ imaginary, we have

$$H^* = a_1\sigma_1 - a_2\sigma_2 + a_3\sigma_3 \ .$$

The top equation ca be translated into relations between $S$ and Pauli matrices: $S^{-1}\sigma_1 S = -\sigma_1$, $S^{-1}\sigma_2 S = \sigma_2$ and $S^{-1}\sigma_3 S = -\sigma_3$. Namely, the matrix $S$ must commute with $\sigma_2$ and anticommute with $\sigma_1$ and $\sigma_3$. This can be satisfied with

$$S = c\sigma_2$$

where $c$ is some arbitrary constant. If we choose $c = 1$, the matrix $S$ is unitary and hermitian; for $c = i$, $S$ is real.

b) The statement '$\psi_1$ and $\psi_2$ are the bases for the spin-$\frac{1}{2}$ representation of $SU(2)$' means that under an $SU(2)$ transformation ($i = 1, 2$)

$$\psi_i \to \psi_i' = U_{ij}\psi_j \text{ with } U = \exp(i\vec{\alpha}\cdot\vec{\sigma}) .$$

In matrix notation, this is $\psi' = U\psi$. The complex conjugate equation is then

$$\psi'^* = U^*\psi^* = (S^{-1}US)\psi^* \text{ or } (S\psi'^*) = U(S\psi^*) .$$

This means that $S\psi^*$ has the same transformation properties as $\psi$. Explicitly, with $S = i\sigma_2$, we have

$$S\psi^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \psi_1^* \\ \psi_2^* \end{pmatrix} = \begin{pmatrix} \psi_2^* \\ -\psi_1^* \end{pmatrix} .$$

To say that it has the same transformation properties as

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

means that, for example,

$$T_3\begin{pmatrix} \psi_2^* \\ -\psi_1^* \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}\begin{pmatrix} \psi_2^* \\ -\psi_1^* \end{pmatrix} .$$

Namely, the eigenvalues of the $T_3$ generators are

$$\begin{aligned} t_3(\psi_2^*) &= t_3(\psi_1) = \frac{1}{2} \\ t_3(\psi_1^*) &= t_3(\psi_2) = -\frac{1}{2} \end{aligned}$$

**Remark**: This shows that the $T = \frac{1}{2}$ representation is equivalent to its complex conjugate representation. We say that it is a **real representation**. This property can be extended to all other representations of the $SU(2)$ group, because all other representations can be obtained from the $T = \frac{1}{2}$ representation by tensor product. Part b) shows that the matrix $S$ transforms any real diagonal matrix, e.g. $\sigma_3$, into the **negative** of itself. In other words, $S$ will transform any eigenvalue to its negative. Thus the existence of such a matrix $S$ requires that the eigenvalues of the hermitian-generating matrix occur in pairs of the form $\pm\alpha_1$, $\pm\alpha_2, \ldots$ (or are zero). It is then clear that for groups $SU(n)$ with $n \geq 3$, such a matrix $S$ cannot exist as eigenvalues of higher-rank special unitary groups do not have such a special pairwise structure.

## Question 4

a) The matrix $J$, defined as $J(\lambda) \equiv e^{i\lambda A}Be^{-i\lambda A}$, begin a function of some real parameter $\lambda$, can be differentiated to yield:

$$\begin{aligned} \frac{dJ}{d\lambda} &= e^{i\lambda A}i[A,B]e^{-i\lambda A} \Rightarrow \left.\frac{dJ}{d\lambda}\right|_{\lambda=0} = i[A,B] \equiv iC_1 \\ \frac{d^2J}{d\lambda^2} &= e^{i\lambda A}i^2[A,[A,B]]e^{-i\lambda A} \Rightarrow \left.\frac{d^2J}{d\lambda^2}\right|_{\lambda=0} = i^2[A,[A,B]] \equiv i^2C_2 \\ &\vdots \qquad \vdots \\ \frac{d^nJ}{d\lambda^n} &= e^{i\lambda A}i^n[A,C_{n-1}]e^{-i\lambda A} \Rightarrow \left.\frac{d^nJ}{d\lambda^n}\right|_{\lambda=0} = i^n[A,C_{n-1}] \equiv i^nC_1 \end{aligned}$$

Expand $J(\lambda)$ in a Taylor series:

$$J(\lambda) = \sum_{n=0}^{\infty} \frac{d^n J}{d\lambda^n}\bigg|_{\lambda=0} \frac{\lambda^n}{n!} = \sum_{n=0}^{\infty} i^n C_n \frac{\lambda^n}{n!}$$

where $C_0 = B$, $C_1 = [A, B]$ and $C_n = [A, C_{n-1}]$. Setting $\lambda = 1$, we have the desired result

$$e^{iA} B e^{-iA} = B + i[A, B] + \frac{i^2}{2!}[A, [A, B]] + \dots \ .$$

b) To show that 'the matrix $B$ is invariant (up to a phase) under transformations generated by matrix $A$' means to show that

$$e^{i\alpha A} B e^{-i\alpha A} = B$$

for an arbitrary real parameter $\alpha$. But from part a) we have already show that

$$e^{i\alpha A} B e^{-i\alpha A} = \sum_{n=0}^{\infty} i^n C_n \frac{\alpha^n}{n!}$$

where $C_0 = B$, $C_1 = [A, B]$ and $C_n = [A, C_{n-1}]$. For the case at hand of $[A, B] = B$ we have $C_n = B$ for all $n = 0, 1, \dots$

$$e^{i\alpha A} B e^{-i\alpha A} = B \sum_{n=0}^{\infty} i^n \frac{\alpha^n}{n!} = B e^{i\alpha} \ .$$

This is the claimed result.

## Question 5

We will first derive a useful identity for Pauli matrices. Consider the multiplication of two matrices

$$\begin{aligned}
(\vec{A} \cdot \vec{\sigma})(\vec{B} \cdot \vec{\sigma}) &= (\sigma_i \sigma_j) A_i B_j \\
&= \frac{1}{2}[(\sigma_i \sigma_j + \sigma_j \sigma_i) + (\sigma_i \sigma_j - \sigma_j \sigma_i)] A_i B_j \\
&= \frac{1}{2}(\{\sigma_i, \sigma_j\} + [\sigma_i, \sigma_j]) A_i B_j \\
&= \frac{1}{2}(2\delta_{ij} + 2i\epsilon_{ijk}\sigma_k) A_i B_j
\end{aligned}$$

where we have used the basic commutation relations satisfied by the Pauli matrices:

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \ \text{ and } \ \{\sigma_i, \sigma_j\} = 2\delta_{ij} \ .$$

Thus we have the identity

$$(\vec{A} \cdot \vec{\sigma})(\vec{B} \cdot \vec{\sigma}) = \vec{A} \cdot \vec{B} + i\vec{\sigma} \cdot (\vec{A} \times \vec{B})$$

Set $\vec{A} = \vec{B} = \vec{r}$, we get $(\vec{r} \cdot \vec{\sigma})^2 = r^2 + i\vec{\sigma} \cdot (\vec{r} \times \vec{r}) = r^2$ and $(\vec{r} \cdot \vec{\sigma})^3 = r^2)\vec{r} \cdot \vec{\sigma}) = r^3(\hat{r} \cdot \vec{\sigma})$. It is then straightforward to see that

$$(\vec{r} \cdot \vec{\sigma})^{2n} = r^{2n} \ \text{ and } \ (\vec{r} \cdot \vec{\sigma})^{2n+1} = r^{2n+1}(\hat{r} \cdot \vec{\sigma})$$

with $n = 1, 2, \dots$. The desired identity for the unitary matrix then follows as

$$\begin{aligned}
\exp(i\vec{r} \cdot \vec{\sigma}) &= \sum_n \frac{i^n}{n!}(\vec{r} \cdot \vec{\sigma})^n \\
&= \sum_{n=even} \frac{i^n}{n!} r^n + (\hat{r} \cdot \vec{\sigma}) \sum_{n=odd} \frac{i^n}{n!} r^n \\
&= \cos r + (\hat{r} \cdot \vec{\sigma}) \sin r \ .
\end{aligned}$$

**Remark**: This relation holds only for $2 \times 2$ unitary matrices and does not hold for higher-dimensional cases, where anticommutation relations are much more complicated than just the Kronecker delta.

## Question 6

From the Schrödinger equation we can multiply by $\psi^*$, that is,

$$i\hbar\psi^* \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\psi^* \nabla^2 \psi + \psi^* V \psi$$

Similarly, if we conjugate the Schrödinger equation and multiply by $\psi$

$$-i\hbar \frac{\partial \psi^*}{\partial t}\psi = -\frac{\hbar^2}{2m}(\nabla^2 \psi^*)\psi + V\psi^*\psi$$

The difference of these two equations yields

$$i\hbar \left(\psi^* \frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t}\psi\right) = -\frac{\hbar^2}{2m}\left(\psi^* \nabla^2 \psi - (\nabla^2 \psi^*)\psi\right)$$

$$\text{or} \quad i\hbar \frac{\partial}{\partial t}(\psi^* \psi) = -\frac{\hbar^2}{2m}\vec{\nabla} \cdot \left(\psi^* \vec{\nabla}\psi - (\vec{\nabla}\psi^*)\psi\right)$$

$$\frac{\partial}{\partial t}(\psi^*\psi) + \frac{\hbar}{2im}\vec{\nabla} \cdot \left(\psi^* \vec{\nabla}\psi - (\vec{\nabla}\psi^*)\psi\right) = 0$$

Defining $j^\mu \equiv (c\rho, \vec{j})$, then $\partial_\mu j^\mu = 0$ means

$$\begin{aligned} c\rho &= c\psi^*\psi \\ \vec{j} &= \frac{\hbar}{2im}\left(\psi^* \vec{\nabla}\psi - (\vec{\nabla}\psi^*)\psi\right) \ . \end{aligned}$$