Physics Honours: Standard Model

Tutorial Sheet 1

Question 1

If an electron has orbital angular momentum equal to $4.714 \times 10^{-34} J \cdot s$, what is the orbital quantum number for the state of the electron.

Question 2

a) Find the mass density of a proton, modeling it as a solid sphere of radius $1.00 \times 10^{-15} m$.

b) **What If?** Consider a classical model of an electron as a solid sphere with the same density as the proton. Find its radius.

c) Imagine that this electron possess spin angular momentum $I\omega = \hbar/2$ because of classical rotation about the z-axis. Determine the speed of a point on the equator of the electron and

d) compare this speed to the speed of light.

Question 3

Particles known as resonances have very short life-times, on the order of $10^{-23} s$. From this information would you guess that they are hadrons or leptons? Explain

Question 4

The Ξ^0 particle decays by the weak interaction according to the decay mode $\Xi^0 \to \Lambda^0 + \pi^0$. Would you expect this decay to be fast or slow? Explain.

Question 5

Two protons in a nucleus interact via the nuclear interaction. Are they also subject to the weak interaction?

Question 6

Recall that under Lorentz transformations

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

a) Obtain the explicit form of the matrix $\Lambda^\mu{}_\nu$ for a "boost" along the x-axis.

b) Obtain the explicit form of the matrix $\Lambda^\mu{}_\nu$ for a "rotation" through an angle θ about the z-axis.

Question 7

Consider an arbitrary rotation in three spatial dimensions

$$X_i \to X'_i = O_{ij} X_j \quad i, j = 1, 2, 3$$

or equivalently, using a matrix notation

$$X \to X' = OX$$

a) Explain why $X_i X_i = X^T X = \text{invariant}$

b) and hence that the matrix O must satisfy

$$O^T O = 1 \quad \Leftrightarrow O^{-1} = O^T$$

c) What matrices are these?

Question 8

Consider now a Lorentz transformation

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

Associate a column vector X to x^μ, so that in matrix form (where Λ is a matrix)

$$X \to X' = \Lambda X$$

a) Explain why for a Lorentz transformation

$$X^T g X = \text{invariant}$$

where $g_{\mu\nu} \to g$ is the metric.

b) Hence show that for a Lorentz transformation

$$\Lambda^T g \Lambda = g$$

c) Show that in vector/tensor/index notation this requires

$$\Lambda^\mu{}_\alpha g_{\mu\nu} \Lambda^\nu{}_\beta = g_{\alpha\beta}$$

Answers

Question 1

$$L = \sqrt{\ell(\ell+1)} \Rightarrow 4.714 \times 10^{-34} = \sqrt{\ell(\ell+1)}\left(\frac{6.626 \times 10^{-34}}{2\pi}\right)$$

$$\begin{aligned} \text{or } \ell(\ell+1) &= \frac{(4.714 \times 10^{-34})^2(2\pi)^2}{(6.626 \times 10^{-34})^2} = 19.98 \approx 20 \\ \Rightarrow \ell &= 4 \end{aligned}$$

Question 2

a) Density of a proton:

$$\rho = \frac{m}{V} = \frac{1.67 \times 10^{-27} kg}{\left(\frac{4}{3}\right)\pi(1.00 \times 10^{-15} m)^3} = 3.99 \times 10^{17} kg/m^3$$

b) Size of model electron:

$$r = \left(\frac{3m}{4\pi\rho}\right)^{1/3} = \left(\frac{3(9.11 \times 10^{-31} kg)}{4\pi(3.99 \times 10^{17} kg/m^3)}\right)^{1/3} = 8.17 \times 10^{-17} m$$

c) Moment of Inertia:

$$\begin{aligned} I &= \frac{2}{5}mr^2 = \frac{2}{5}(9.11 \times 10^{-31} kg)(8.17 \times 10^{-17} m)^2 \\ &= 2.43 \times 10^{-63} kg \cdot m^2 \\ L_z &= I\omega = \frac{\hbar}{2} = \frac{Iv}{r} \\ \text{Therefore } v &= \frac{\hbar r}{2I} = \frac{(6.626 \times 10^{-34} J \cdot s)(8.17 \times 10^{-17} m)}{2\pi(2 \times 2.43 \times 10^{-63} kg \cdot m^2)} \\ &= 1.77 \times 10^{12} m/s \end{aligned}$$

d) This is 5.91×10^3 times larger than the speed of light.

Question 3

Resonances are hadrons. They decay into strongly interacting particles such as protons, neutrons and pions, all of which are hadrons.

Question 4

Decays by the weak interaction typically take $10^{-10} s$ or longer to occur. This is slow in particle physics.

Question 5

Yes, protons interact via the weak interaction; but the strong interaction predominates.

Question 6

a) As learnt in previous studies

$$\begin{array}{c} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ t' = \gamma\left(t - \frac{v}{c^2}x\right) \end{array} \Rightarrow \left\{ \begin{array}{c} ct' = \left(ct - \frac{v}{c}x\right)\gamma \\ x' = \left(x - \frac{v}{c}(ct)\right)\gamma \\ y' = y \\ z' = z \end{array} \right.$$

$$\text{Therefore} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{v\gamma}{c} & 0 & 0 \\ -\frac{v\gamma}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

b) For rotations about the z-axis through an angle θ

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\text{adding } ct\ ; \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

Question 7

a) Rotations preserve length, $(length)^2 = \sum_i X_i^2 = X_i X_i$

$$X_i X_i = (X_1\ X_2\ \ldots\ X_N) \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{pmatrix} = X^T X \tag{1}$$

using vector notation.

b) Suppose $X \to X' = OX$ ($X'_i = O_{ij} X_j$). Then $X^T \to X'^T = (OX)^T = X^T O^T$ ($X'_i = X_j (O^T)_{ji}$).
Do not forget $(AB)^T = B^T A^T$ as $(AB)_{ij} = A_{ik} B_{kj}$ and $((AB)^T)_{ij} = (AB)_{ji} = A_{jk} B_{ki} = B_{ki} A_{jk} = (B^T)_{ik} (A^T)_{kj} = (B^T A^T)_{ij}$.
Therefore $X'_i X'_i = X'^T X' = X^T O^T O X = X^T X \Rightarrow O^T O = 1 \Rightarrow O^T = O^{-1}$.

c) **Orthonormal matrices** (which are a subgroup of **real** unitary matrices)

Question 8

a) We have $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$ and we can replace $x^{\mu} = \begin{pmatrix} x^0 \\ \vdots \\ x^3 \end{pmatrix}$ with X and $\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \Lambda^1_1 & \Lambda^1_2 & \ldots \\ & \cdots & \\ & \vdots & \end{pmatrix}$ with Λ. In which case $X' = \Lambda X$.
For a Lorentz transformation $x^{\mu} x_{\mu} = \text{invariant} = x^{\mu} g_{\mu\nu} x^{\nu} \equiv X^T g X$ where in matrix notation $g_{\mu\nu} \to g$

b) In which case $x'_{\mu} x'_{\mu} = x^{\mu} x_{\mu} \Rightarrow X'^T g X' = X^T g X$. If $X' = \Lambda X$ then $(X')^T = (\Lambda X)^T = X^T \Lambda^T$. Therefore $X'^T g X' = X^T \Lambda^T g \Lambda X = X^T g X \Rightarrow \Lambda^T g \Lambda = g$.

c) As $g \to g_{\mu\nu}$ and $\Lambda \to \Lambda^{\mu}{}_{\nu}$ therefore Λ^T must have indices $(\Lambda^T)_{\mu}{}^{\nu}$. In which case $g = g_{\mu\nu} = \Lambda^T g \Lambda = (\Lambda^T)_{\mu}{}^{\alpha} g_{\alpha\beta} \Lambda^{\beta}{}_{\nu} = \Lambda^{\alpha}{}_{\mu} g_{\alpha\beta} \Lambda^{\beta}{}_{\nu}$ as by the definition of $(\Lambda^T)_{\mu}{}^{\nu} = \Lambda^{\nu}{}_{\mu}$. Swapping indices $g_{\alpha\beta} = \Lambda^{\mu}{}_{\alpha} g_{\mu\nu} \Lambda^{\nu}{}_{\beta}$.