Physics Honours: Standard Model

Tutorial Sheet 2

Question 1

Kaons all decay into final states that contain no protons or neutrons. What is the baryon number of kaons?

Question 2

An antibaryon interacts with a meson. Can a baryon be produced in such an interaction? Explain

Question 3

Each of the following reactions is forbidden. Determine a conservation law that is violated for each reaction

a) $p + \bar{p} \rightarrow \mu^+ + e^-$

b) $p + \pi^- \rightarrow \pi^+ + p$

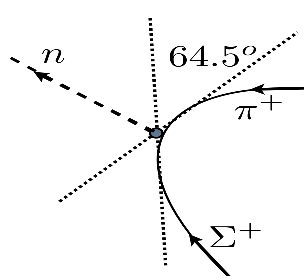
c) $p + p \rightarrow \pi^+ + p$

d) $p + p \rightarrow p + p + n$

e) $p + \gamma \rightarrow \pi^0 + n$

Question 4

The particle decay $\Sigma^+ \rightarrow \pi^+ + n$ is observed in a bubble chamber. The figure below represents the curved tracks of the particles Σ^+ and π^+, and the invisible track of the neutron, in the presence of a uniform magnetic field of $1.15T$ directed out of the page. The measured radii of curvature are $1.99m$ for the Σ^+ particle and $0.580m$ for the π^+ particle



a) Find the momenta of the Σ^+ and the π^+ particles, in units of MeV/c

b) The angle between the momenta of the Σ^+ and the π^+ particles at the moment of decay is 64.5^o. Find the momentum of the neutron

c) Calculate the total energy of the π^+ particle, and of the neutron, from their known masses ($m_\pi = 139.6MeV/c^2$, $m_n = 939.6MeV/c^2$) and the relativistic energy-momentum relation. What is the total energy of the Σ^+ particle

d) Calculate the mass and speed of the Σ^+ particle.

Question 5

Consider an infinitesimal boost along the x-axis (that is, $v/c \ll 1$). Then

$$a^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \epsilon^{\mu}{}_{\nu}$$

Obtain the form of $\epsilon^{\mu}{}_{\nu}$ and $\epsilon_{\mu\nu}$.

Question 6

Consider a relativistic equation for a free relativistic particle based on the relationship

$$\begin{aligned} E &= \sqrt{(mc^2)^2 + p^2c^2} \\ \hat{E}\phi(\vec{r},t) &= \sqrt{(mc^2)^2 + c^2\hat{p}^2}\phi(\vec{r},t) \end{aligned} \tag{1}$$

Show that the solutions of the equation which are eigenstates of energy and momentum are

$$\phi_{\vec{k}}(\vec{r},t) = \frac{1}{(2\pi)^{3/2}} e^{-i(\omega_{\vec{k}}t - \vec{k}\cdot\vec{r})} \ ; \ \omega_{\vec{k}} = \sqrt{c^2k^2 + \left(\frac{mc^2}{k}\right)^2}$$

and hence that the most general solution to equation (1) is given by

$$\phi(\vec{r},t) = \int \frac{d^3k}{(2\pi)^{3/2}} f_{\vec{k}} e^{-i(\omega_{\vec{k}}t - \vec{k}\cdot\vec{r})} \quad \text{with} \quad f_{\vec{k}} = \int \frac{d^3\vec{r}}{(2\pi)^{3/2}} e^{i(\omega_{\vec{k}}t - \vec{k}\cdot\vec{r})} \phi(\vec{r},t)\ .$$

Question 7

The Klein paradox. Recall that the Klein-Gordon equation of a particle moving in the presence of a potential $V = q\phi$ in one dimension is given by

$$\left(i\hbar\frac{\partial}{\partial t} - V\right)^2 \phi = \left[-c^2\hbar^2\frac{\partial^2}{\partial x^2} + m^2c^4\right]\phi\ .$$

Consider a particle of energy $E > mc^2$ incident on a potential barrier

$$V(x) = \begin{cases} V & x > 0 \\ 0 & x < 0 \end{cases}$$

a) Writing $\phi(x,t) = e^{-iEt/\hbar}\phi_E(x)$, show that

$$(E - V)^2\phi_E(x) = \left[-c^2\hbar^2\frac{d^2}{dx^2} + m^2c^4\right]\phi_E(x)\ .$$

b) Show that the solution to this equation can be written as

$$\begin{aligned} \phi_E(x) &= Ae^{ikx} + Be^{-ikx} \qquad x < 0\ ; \quad E^2 = (c\hbar k)^2 + m^2c^4 \\ \phi_E(x) &= Ce^{i\gamma x} \qquad x > 0 \ ; \quad (E - V)^2 = (c\hbar\gamma)^2 + m^2c^4 \end{aligned}$$

c) Requiring continuity of $\phi_E(x)$ and $\frac{d\phi_E}{dx}(x)$ at $x = 0$, show that

$$\frac{B}{A} = \frac{k - \gamma}{k + \gamma} \quad ; \quad \frac{C}{A} = \frac{2k}{k + \gamma}$$

d) Consider first the case when

$$V < E < mc^2 + V$$

Show that in this case the solution for $x > 0$ is given by

$$\phi_E(x) = De^{-\alpha x} \qquad ; \qquad \alpha = \frac{1}{\hbar c}\sqrt{(mc^2)^2 - (E-V)^2}$$

and that the charge density in this region is given by

$$\rho(x) = \frac{E-V}{mc^2}|D|^2 e^{-2\alpha x}$$

(see note at the end of the question). Hence argue why the particle is localised insider the barrier within a distance

$$\sim \hbar c / 2\sqrt{(mc^2)^2 - (E-V)^2} \ ,$$

and compare this result with the non-relativistic case.

e) In an attempt to localise further the particle, V is increased so that

$$V - mc^2 < E < V \ .$$

Show that the solution of d) is still valid, but now $\rho(x) < 0$!

f) In an effort to localise even further the particle in the barrier, the potential is further increased so that

$$V > E + mc^2 \ .$$

i) Show that γ is real again, that is, there is again a particle current in this region $x > 0$!

ii) Show that the group velocity of the current is

$$v_g = \frac{\hbar c - \gamma}{E - V}$$

and hence that in order for there to be a wave packet moving to the right $\gamma < 0$.

iii) Show that when $\gamma < 0$, the reflection coefficient B/A is larger than one, that is, more wave is reflected than is incident!

g) Any explanation for the somewhat unusual features of the answers to e) and f)?

Note: Because the particle is coupled to the electromagnetic field (only Coulomb potential in this case), we need to use minimal coupling

$$i\hbar\frac{\partial}{\partial t} \to i\hbar\frac{\partial}{\partial t} - q\phi = i\hbar\frac{\partial}{\partial t} - V$$
$$\Rightarrow \frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \frac{i}{\hbar}V$$

Answers

Question 1

The baryon number of a proton or neutron is one. Since baryon number is conserved, the baryon number of the kaon must be zero.

Question 2

No. Antibaryons have baryon number -1, meson have baryon number 0, and baryons have baryon number $+1$. The reaction cannot occur because it would not conserve baryon number, unless so much energy is available that a baryon-antibaryon pair is produced.

Question 3

a) $p + \bar{p} \to \mu^+ + e^-$: L_e numbers are $0 + 0 \to 0 + 1$, whilst L_μ we have $0 + 0 \to -1 + 0$

b) $p + \pi^- \to \pi^+ + p$: for charge we have $-1 + 1 \to +1 + 1$

c) $p + p \to p + \pi^+$: baryon numbers are $1 + 1 \to 1 + 0$

d) $p + p \to p + p + n$: baryon numbers are $1 + 1 \to 1 + 1 + 1$

e) $p + \gamma \to \pi^0 + n$: for charge we have $0 + 1 \to 0 + 0$

Question 4

a)

$$\begin{aligned} p_{\Sigma^+} &= eBr_{\Sigma^+} = \frac{(1.602177 \times 10^{-19}C)(1.15T)(1.99m)}{5.344288 \times 10^{-22}(kg \cdot m/s)/(MeV/c)} \\ &= 686MeV/c \\ p_{\pi^+} &= eBr_{\pi^+} = \frac{(1.602177 \times 10^{-19}C)(1.15T)(0.580m)}{5.344288 \times 10^{-22}(kg \cdot m/s)/(MeV/c)} \\ &= 200MeV/c \end{aligned}$$

b) Let φ be the angle made by the neutron's path with the path of the Σ^+ at the moment of decay. By conservation of momentum

$$\begin{aligned} p_n \cos\varphi + (199.961581MeV/c)\cos 64.5^o &= 686.075081MeV/c \\ p_n \cos\varphi &= 599.989401MeV/c \\ p_n \sin\varphi = (199.961581MeV/c)\sin 64.5^o &= 180.482380MeV/c \\ \Rightarrow p_n &= \sqrt{(599.989401MeV/c)^2 + (180.482380MeV/c)^2} \\ &= 627MeV/c \end{aligned}$$

c)

$$\begin{aligned} E_{\pi^+} &= \sqrt{(p_{\pi^+}c)^2 + (m_{\pi^+}c^2)^2} = \sqrt{(199.961581MeV)^2 + (139.6MeV)^2} \\ &= 244MeV \\ E_n &= \sqrt{(p_n c)^2 + (m_n c^2)^2} = \sqrt{(626.547022MeV)^2 + (939.6MeV)^2} \\ &= 1130MeV \\ E_{\Sigma^+} &= E_{\pi^+} + E_n = 1370MeV \end{aligned}$$

d)

$$\begin{aligned} m_{\Sigma+}c^2 &= \sqrt{E_{\Sigma+}^2 - (p_{\Sigma+}c)^2} = \sqrt{(1373.210664MeV)^2 - (686.075081MeV)^2} \\ &= 1190MeV \end{aligned}$$

$$\Rightarrow m_{\Sigma+} = 1190MeV/c^2$$

$$E_{\Sigma+} = \gamma m_{\Sigma+}c^2 \text{ where } \gamma = \sqrt{1-\frac{v^2}{c^2}} = \frac{1373.210664MeV}{1189.541303MeV} = 1.1544. \text{ Solving for } v,\ v = 0.500c.$$

Question 5

From Question 6 a) of tutorial sheet 1 we recall that

$$a^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\gamma = \left(1-\frac{v^2}{c^2}\right)^{-1/2} = (1-\beta^2)^{-1/2} \approx 1 + \frac{1}{2}\beta^2 + \ldots = 1 + \mathcal{O}(\beta^2)$. Therefore, to order $\beta = v/c$

$$\begin{aligned} a^\mu{}_\nu &= \begin{pmatrix} 1 & -\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\beta & 0 & 0 \\ -\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ a^\mu{}_\nu &\approx \delta^\mu{}_\nu + \epsilon^\mu{}_\nu \end{aligned}$$

$$\epsilon_{\mu\nu} \equiv g_{\mu\alpha}\epsilon^\alpha{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -\beta & 0 & 0 \\ -\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\beta & 0 & 0 \\ \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Note that this is antisymmetric.

Question 6

$\phi_{\vec{k}}(\vec{r},t) = \frac{1}{(2\pi)^{3/2}} e^{-i(\omega_{\vec{k}}t - \vec{k}\cdot\vec{r})}$ is an eigenfunction of energy and momentum (plane wave) satisfying

$$\begin{aligned} \hat{E}\phi_{\vec{k}}(\vec{r},t) &= i\hbar\frac{\partial}{\partial t}\phi_{\vec{k}}(\vec{r},t) = \hbar\omega_{\vec{k}}\phi_{\vec{k}}(\vec{r},t) \\ \hat{\vec{p}}\phi_{\vec{k}}(\vec{r},t) &= \frac{\hbar}{i}\frac{\partial}{\partial x^i}\phi_{\vec{k}}(\vec{r},t) = \hbar\vec{k}\phi_{\vec{k}}(\vec{r},t) \end{aligned}$$

Therefore $\hat{p}^2\phi_{\vec{k}}(\vec{r},t) = (\hbar^2k^2)\phi_{\vec{k}}(\vec{r},t)$ and in general $\hat{p}^{2n}\phi_{\vec{k}}(\vec{r},t) = (\hbar^2k^2)^n\phi_{\vec{k}}(\vec{r},t)$.
Therefore $\sqrt{(mc^2)^2 + c^2\hat{p}^2}\phi_{\vec{k}}(\vec{r},t)$ can be defined by a Taylor expansion, and

$$i\hbar\frac{\partial}{\partial t}\phi_{\vec{k}}(\vec{r},t) = \hbar\omega_{\vec{k}}\phi_{\vec{k}}(\vec{r},t) = \sqrt{(mc^2)^2 - c^2\hbar^2\nabla^2}\phi_{\vec{k}}(\vec{r},t) = \sqrt{(mc^2)^2 + c^2\hbar^2k^2}\phi_{\vec{k}}(\vec{r},t)$$

Therefore, $\hbar\omega_{\vec{k}} = \sqrt{(mc^2)^2 + c^2\hbar^2k^2} \Rightarrow \omega_{\vec{k}} = \sqrt{c^2k^2 + \left(\frac{mc^2}{k}\right)^2}$.
So the most general solution is a linear combination

$$\phi(\vec{r},t) = \int d^3k f_{\vec{k}}\phi_{\vec{k}}(\vec{r},t) = \int \frac{d^3k}{(2\pi)^{3/2}} f_{\vec{k}} e^{-i(\omega_{\vec{k}}t - \vec{k}\cdot\vec{r})}$$

Question 7

a)

$$\left(i\hbar\frac{\partial}{\partial t} - V\right)^2 e^{-iEt/\hbar}\phi_E(x) = \left[-c^2\hbar^2\frac{\partial^2}{\partial x^2} + m^2c^4\right] e^{-iEt/\hbar}\phi_E(x)$$
$$\Rightarrow (E-V)^2\phi_E(x) = \left[-c^2\hbar^2\frac{d^2}{dx^2} + m^2c^4\right]\phi_E(x)$$

b) For $x < 0$, $V = 0 \Rightarrow \phi_E = e^{\pm ikx}$ with $E^2 = \hbar^2c^2k^2 + m^2c^4$
For $x > 0$, $\phi_E = e^{i\gamma x}$ with $(E-V)^2 = \hbar^2c^2\gamma^2 + m^2c^4$
with other solutions not being physical.

c) From continuity of ϕ_E we have $A+B = C$, and from continuity of $\frac{d\phi_E}{dx}$ we have $k(A-B) = \gamma C$. Rearranging $A - B = \frac{\gamma}{k}C$ which when added with our first equation give $2A = \left(1+\frac{\gamma}{k}\right)C = \left(\frac{k+\gamma}{k}\right)C \Rightarrow \frac{C}{A} = \frac{2k}{k+\gamma}$.
If we were to subtract our first and a rearranged second equation $((A-B)k/\gamma = C) \Rightarrow A\left(1-\frac{k}{\gamma}\right) + B\left(1+\frac{k}{\gamma}\right)$ which leads to $\frac{B}{A} = \frac{k-\gamma}{k+\gamma}$.

d) $(E-V)^2 - m^2c^4 = (c\hbar\gamma)^2$, now since $E - V < mc^2 \Rightarrow \gamma$ is imaginary. So let us call $\gamma = i\alpha$ in which case $\alpha = \frac{1}{c\hbar}\sqrt{(mc^2)^2 - (E-V)^2}$ and $\phi_E(x) = De^{-\alpha x}$.
Note that $\rho(x) = \frac{i\hbar}{2mc^2}\left[\phi^\dagger\left(\frac{\partial}{\partial t} + \frac{iV}{\hbar}\right)\phi - \phi\left(\frac{\partial}{\partial t} - \frac{iV}{\hbar}\right)\phi^\dagger\right]$

$$\text{Now } \left(\frac{\partial}{\partial t} + \frac{iV}{\hbar}\right)De^{-iEt/\hbar}e^{-\alpha x} = \left(-\frac{iE}{k} + \frac{iV}{\hbar}\right)De^{-iEt/\hbar}e^{-\alpha x} = -\frac{i}{\hbar}(E-V)\phi$$
$$\left(\frac{\partial}{\partial t} - \frac{iV}{\hbar}\right)\phi^\dagger = \frac{i}{\hbar}(E-V)\phi \Rightarrow$$
$$\rho(x) = \frac{i\hbar}{2mc^2}\left[-\frac{i}{\hbar}(E-V)\phi^\dagger\phi - \frac{i}{\hbar}(E-V)\phi^\dagger\phi\right] = \frac{E-V}{2mc^2}|D|^2e^{-2\alpha x}$$

Now as distance $\sim \frac{1}{2\alpha} \sim \frac{\hbar c}{2\sqrt{(mc^2)^2 - (E-V)^2}}$. Which is similar to the non-relativistic case.

e) $V - E < mc^2$ so $\gamma = i\alpha$ is imaginary, but $E - V < 0$, so from part d), $\rho(x) < 0$!

f) i) $V > E + mc^2$ or $V - E > mc^2 \Rightarrow \gamma$ is now real and $\phi_E = e^{i\gamma x}$ for $x \geq 0$.
ii) So we have $(E-V)^2 = \hbar^2c^2\gamma^2 + m^2c^4 \Rightarrow 2(E-V)dE = 2\hbar^2c^2\gamma d\gamma$. The group velocity is

$$v_g = \frac{d\omega_\gamma}{d\gamma} = \frac{d(\hbar\omega_\gamma)}{d(\hbar\gamma)} = \frac{dE}{\hbar d\gamma} = \frac{\hbar c^2\gamma}{E-V}$$

So if $v_g > 0$, and since $E - V < 0$, $\gamma < 0$ that is, negative.

iii)

$$\frac{B}{A} = \frac{k-\gamma}{k+\gamma} = \frac{k-|\gamma|}{k+|\gamma|} > 1$$

g) Sufficient energy to create antiparticles in the barrier, which move to the left.