Physics Honours: Standard Model

Tutorial Sheet 4

Question 1

Consider the transformation of the 4-vector $\left(1, \vec{0}\right)$ under a boost in the $\hat{z}$ direction:

$$\left(1, \vec{0}\right) \rightarrow \left(\gamma, -\sqrt{\gamma^2 - 1}\hat{z}\right) \; ,$$

where $\gamma = \sqrt{v^2 - 1}$ parameterises the boost.

Writing $\gamma = \cosh\phi$ show that the boost corresponds to the transformation matrix $Q = \exp\left(\sigma_3 \phi/2\right)$ (note that there is no i in our matrix representation).

Question 2

Consider the Dirac representation of the α_i and β matrices given in class. Verify explicitly (using 2×2 block notation) that

i) α_i and β are Hermitian

ii) $\alpha_i^2 = \beta^2 = 1$

iii) $\{\alpha_i, \alpha_j\} = 2\delta_{ij}$

iv) $\{\beta, \alpha_i\} = 0$

Question 3

From the definitions of the gamma matrices γ^μ, and the properties from question 2 for the α_i's and β, show that

$$\gamma^{i\,\dagger} = -\gamma^i \qquad \gamma^{0\,\dagger} = \gamma^0 \qquad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

Question 4

Define the "gamma five" matrix γ_5 to be

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

Show that $\gamma_5^2 = 1$, $\{\gamma_5, \gamma^\mu\} = 0$, and obtain an explicit representation of γ_5 for both the Dirac and chiral representations.

Question 5

From lectures we have $\langle 0 \,| u_+(0) |\, k\rangle \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, show that $J_z|k\rangle = \lambda|k\rangle$ where $\lambda = \frac{1}{2}$ by considering how $u_+(0)$ transforms under rotations about the z-axis by an angle θ.

Note that you can similarly show for the spinor $v_+ \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ multiplying a creation operator creates a state with angular momentum $-\frac{1}{2}$ along the direction of motion.

Question 6

The Dirac spinor in momentum space can be written as:

$$u(p, \pm) = \sqrt{2m} \begin{pmatrix} 1 \\ \dfrac{\vec{\sigma} \cdot \vec{p}}{E + m} \end{pmatrix} \chi_\pm \ ,$$

where $(\vec{\sigma} \cdot \hat{p})\, \chi_\pm = \pm \chi_\pm$ with $\hat{p} = \vec{p}/|\vec{p}|$.

Show that the left-handed and right-handed spinors given by:

$$u_L(p) = \frac{1}{2} (1 - \gamma_5)\, u(p, -) \ , \quad u_R(p) = \frac{1}{2} (1 + \gamma_5)\, u(p, +) \ ,$$

are eigenstates of the helicity operator $\lambda = \vec{s} \cdot \vec{p}$ in the massless limit, where the spin operator is of the form

$$\vec{s} = \frac{1}{2} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \ .$$

Note that the same calculation should also show that the other two combinations:

$$\frac{1}{2} (1 + \gamma_5)\, u(p, -) \ , \quad \frac{1}{2} (1 - \gamma_5)\, u(p, +) \ ,$$

are identically zero in the same limit

Question 7

Recall that for an infinitesimal Lorentz transformation the Dirac spinor transforms according to:

$$\begin{aligned} \psi(x) \;\to\; \psi'(x') = S\psi(x) \ \ \text{where} \\ S \;=\; 1 - \frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu} + \ldots \\ \text{and}\epsilon^{\nu}{}_{\mu}\gamma^{\mu} \;=\; -\frac{i}{4}\epsilon^{\alpha\beta}\left[\gamma^{\nu}, \sigma_{\alpha\beta}\right] \end{aligned}$$

Show that this implies $2i\left[\delta^{\nu}_{\alpha}\gamma_{\beta} - \delta^{\nu}_{\beta}\gamma_{\alpha}\right] = [\gamma^{\nu}, \sigma_{\alpha\beta}]$.

Question 8

Show that $P_R^2 = P_R$, $P_L^2 = P_L$, $P_R P_L = 0$ and $P_R + P_L = 1$. Also, show that if $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ then

$$\left[\gamma^\alpha\gamma^\beta, \gamma^\mu\gamma^\nu\right] = 2\eta^{\beta\mu}\gamma^\alpha\gamma^\nu - 2\eta^{\alpha\mu}\gamma^\beta\gamma^\nu + 2\eta^{\beta\nu}\gamma^\mu\gamma^\alpha - 2\eta^{\alpha\nu}\gamma^\mu\gamma^\beta$$

Question 9

The Weyl representation of the Clifford algebra is given by:

$$\gamma_W^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} , \quad \gamma_W^i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} .$$

Find a unitary matrix U such that

$$\gamma_D^\mu = U \gamma_W^\mu U^\dagger ,$$

where γ_D^μ form the Dirac representation of the Clifford algebra:

$$\gamma_D^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} , \quad \gamma_D^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} .$$

Question 10

For a particle described by a spinor $u(p, \lambda)$ we can define the polarisation four-vector $s_\mu(p, \lambda)$ as:

$$s_\mu(p, \lambda) = \frac{1}{2m} \bar{u}(p, \lambda) \gamma_\mu \gamma_5 u(p, \lambda) .$$

a) Show that $s \cdot p = 0$.

b) Calculate s_μ for the particle at rest ($\vec{p} = 0$), with $\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

c) Show that $s^2 = -1$.

d) Suppose for a particle at rest the polarisation vector is given by:

$$s^\mu = (0, \vec{\eta}) \quad \text{with} \quad \vec{\eta}^2 = 1 .$$

Show that in the frame where the particle moves with momentum $\vec{p}$, the spin vector s^μ is given by:

$$s^0 = \frac{\vec{\eta} \cdot \vec{p}}{m} , \quad \vec{s} = \vec{\eta} + \frac{\vec{p}(\vec{\eta} \cdot \vec{p})}{(E + m)m} .$$

Answers

Question 1

Our vector transforms in matrix form are:

$$\begin{aligned} X' &= QXQ^\dagger \\ \begin{pmatrix} \cosh\phi + \sinh\phi & 0 \\ 0 & \cosh\phi - \sinh\phi \end{pmatrix} &= \begin{pmatrix} e^{\phi/2} & 0 \\ 0 & e^{-\phi/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\phi/2} & 0 \\ 0 & e^{-\phi/2} \end{pmatrix} \\ &= \begin{pmatrix} e^{\phi} & 0 \\ 0 & e^{-\phi} \end{pmatrix} \end{aligned}$$

as $Q = Q^\dagger$ and

$$\begin{aligned} e^{\phi} &= \cosh\phi + \sinh\phi \\ e^{-\phi} &= \cosh\phi - \sinh\phi \end{aligned}$$

which shows what we had wanted to show.

Question 2

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \ , \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \quad \sigma_i^2 = 1 \ , \quad \sigma_i^\dagger = \sigma_i$$

i)

$$\begin{aligned} \alpha_i^\dagger &= \begin{pmatrix} 0 & \sigma_i^\dagger \\ \sigma_i^\dagger & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} = \alpha_i \\ \beta^\dagger &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \beta \end{aligned}$$

ii)

$$\begin{aligned} \alpha_i^2 &= \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} = \begin{pmatrix} \sigma_i^2 & 0 \\ 0 & \sigma_i^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1_4 \\ \beta^2 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1_4 \end{aligned}$$

iii)

$$\begin{aligned} \alpha_i\alpha_j + \alpha_j\alpha_i &= \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \\ &= \begin{pmatrix} \sigma_i\sigma_j + \sigma_j\sigma_i & 0 \\ 0 & \sigma_i\sigma_j + \sigma_j\sigma_i \end{pmatrix} \\ \text{but } \sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k &\Rightarrow \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij} \\ \Rightarrow \alpha_i\alpha_j + \alpha_j\alpha_i &= \begin{pmatrix} 2\delta_{ij} & 0 \\ 0 & 2\delta_{ij} \end{pmatrix} = 2\delta_{ij}1_4 \end{aligned}$$

iv)

$$\begin{aligned} \beta\alpha_i + \alpha_i\beta &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = 0 \end{aligned}$$

Question 3

$$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \qquad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma^{i\,\dagger} = \begin{pmatrix} 0 & \sigma_i^\dagger \\ -\sigma_i\dagger & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} = -\gamma^i \qquad \gamma^{0\,\dagger} = \beta^\dagger = \beta = \gamma^0$$

$$\{\gamma^0, \gamma^0\} = 2(\gamma^0)^2 = 2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2 \cdot 1_4$$

$$\begin{aligned}
\{\gamma^0, \gamma^i\} &= \gamma^0\gamma^i + \gamma^i\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\sigma_i \\ -\sigma_i & 0 \end{pmatrix} = 0 \\
\{\gamma^i, \gamma^j\} &= \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_i\sigma_j - \sigma_j\sigma_i & 0 \\ 0 & -\sigma_i\sigma_j - \sigma_j\sigma_i \end{pmatrix} \\
&= -2\delta_{ij}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -2\delta_{ij}1_4
\end{aligned}$$

Therefore $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$.

Question 4

$$\begin{aligned}
\gamma_5 &= i\gamma^0\gamma^1\gamma^2\gamma^3 \\
(\gamma_5)^2 &= (-)\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = (+)(\gamma^0)^2\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 \\
&= (+)(\gamma^1)^2\gamma^2\gamma^3\gamma^2\gamma^3 = (-)\gamma^2\gamma^3\gamma^2\gamma^3 = (+)(\gamma^2)^2(\gamma^3)^2 = (-1)^2 = 1_4 \\
\{\gamma^0, \gamma_5\} &= i\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 + i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 = i\gamma^1\gamma^2\gamma^3 + i(-)^3(\gamma^0)^2\gamma^1\gamma^2\gamma^3 = 0 \\
\{\gamma^1, \gamma_5\} &= i\gamma^1\gamma^0\gamma^1\gamma^2\gamma^3 + i\gamma^1\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1 = -i\gamma^0(\gamma^1)^2\gamma^2\gamma^3 + i(-)^2\gamma^0(\gamma^1)^2\gamma^2\gamma^3 = 0
\end{aligned}$$

Similarly for γ^2 and γ^3, which gives us $\{\gamma^\mu, \gamma_5\} = 0$.

Question 5

We know that rotations are given by:

$$U_R(\hat{z}, \theta)|k\rangle = e^{-i\theta/2}|k\rangle \quad U_R(\hat{z}, \theta)|0\rangle = |0\rangle$$

and that a spinor transforms as:

$$|\psi\rangle \to U_R(\hat{e}, \theta)|\psi\rangle = \exp\left(-i\mathbf{J} \cdot \hat{e}\theta\right)|\psi\rangle = \exp\left(-i\vec{\sigma} \cdot \hat{e}\theta\right)|\psi\rangle$$

So $U_R^\dagger(\hat{z}, \theta)u_+(0)U_R(\hat{z}, \theta) = \exp\left(-i\sigma_3\theta/2\right)u_+(0)$, that is:

$$\begin{aligned}
\langle 0|U_R^\dagger u_+(0)U_R|k\rangle &= \langle 0|e^{-i\sigma_3\theta/2}u_+(0)|k\rangle \\
&\propto e^{-i\sigma_3\theta/2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{-i\theta/2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{-i\lambda\theta}\begin{pmatrix} 1 \\ 0 \end{pmatrix}
\end{aligned}$$

where we recall that $J_z = \sigma_3/2$, in which case $\lambda = \frac{1}{2}$.

Question 6

In the standard (Dirac) representation, we have

$$\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Thus

$$\begin{aligned} u_L(p) &= \frac{1}{2}(1-\gamma_5)u(p,-) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix}\chi_- \\ &= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \frac{-p}{E} \end{pmatrix}\chi_- = \frac{1}{2}\left(\frac{E+p}{E}\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix}\chi_- \\ &= \begin{pmatrix} 1 \\ -1 \end{pmatrix}\chi_- \end{aligned}$$

where we have used $E = p$ for the massless particle. Similarly

$$u_R(p) = \frac{1}{2}(1+\gamma_5)u(p,+) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\chi_+ .$$

Then

$$\begin{aligned} \lambda u_L(p) &= \frac{1}{2}\begin{pmatrix} \vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\chi_- \\ &= \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\vec{\sigma}\cdot\hat{p}\chi_- = -\frac{1}{2}u_L(p) . \end{aligned}$$

Similarly, we have

$$\lambda u_R(p) = \frac{1}{2}u_R(p) .$$

Question 7

$$\begin{aligned} \epsilon^{\nu}{}_{\mu}\gamma^{\mu} &= \epsilon^{\nu\mu}\gamma_{\mu} = \frac{1}{2}\left(\epsilon^{\nu\mu}\gamma_{\mu} - \epsilon^{\mu\nu}\gamma_{\mu}\right) \\ &= \frac{1}{2}\left(\epsilon^{\alpha\beta}\delta^{\nu}_{\alpha}\gamma_{\beta} - \epsilon^{\alpha\beta}\delta^{\nu}_{\beta}\gamma_{\alpha}\right) \\ \text{Therefore } \frac{1}{2}\epsilon^{\alpha\beta}\left(\delta^{\nu}_{\alpha}\gamma_{\beta} - \delta^{\nu}_{\beta}\gamma_{\alpha}\right) &= -\frac{i}{4}\epsilon^{\alpha\beta}\left[\gamma^{\nu}, \sigma_{\alpha\beta}\right] \\ \Rightarrow 2i\left(\delta^{\nu}_{\alpha}\gamma_{\beta} - \delta^{\nu}_{\beta}\gamma_{\alpha}\right) &= \left[\gamma^{\nu}, \sigma_{\alpha\beta}\right] \end{aligned}$$

Note: $[A,[B,C]] = \{\{A,B\},C\} - \{\{A,C\},B\}$, therefore

$$\begin{aligned} \frac{i}{2}\left[\gamma^{\nu}, [\gamma_{\alpha}, \gamma_{\beta}]\right] &= \frac{i}{2}\left\{\{\gamma^{\nu}, \gamma_{\alpha}\}, \gamma_{\beta}\right\} - \frac{i}{2}\left\{\{\gamma^{\nu}, \gamma_{\beta}\}, \gamma_{\alpha}\right\} \\ &= \frac{i}{2}\left\{2\delta^{\nu}_{\alpha}, \gamma_{\beta}\right\} - \frac{i}{2}\left\{2\delta^{\nu}_{\beta}, \gamma_{\alpha}\right\} \\ &= 2i\left(\delta^{\nu}_{\alpha}\gamma_{\beta} - \delta^{\nu}_{\beta}\gamma_{\alpha}\right) \end{aligned}$$

Question 8

$$\begin{aligned} P_R^2 &= \frac{1}{4}(1+\gamma_5)(1+\gamma_5) = \frac{1}{4}(1+2\gamma_5+\gamma_5^2) \\ &= \frac{1}{2}(1+\gamma_5) = P_R \end{aligned}$$

$$
\begin{aligned}
P_L^2 &= \frac{1}{4}(1-\gamma_5)(1-\gamma_5) = \frac{1}{4}(1-2\gamma_5+\gamma_5^2) \\
&= \frac{1}{2}(1-\gamma_5) = P_L \\
P_R P_L &= \frac{1}{4}(1+\gamma_5)(1-\gamma_5) = \frac{1}{4}(1-\gamma_5^2) = 0 \\
P_R + P_L &= \frac{1}{2}(1+\gamma_5+1-\gamma_5) = 1
\end{aligned}
$$

and finally

$$
\begin{aligned}
\left[\gamma^\alpha\gamma^\beta, \gamma^\mu\gamma^\nu\right] &= \gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu - \gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta = \gamma^\alpha\left(2\eta^{\beta\mu} - \gamma^\mu\gamma^\beta\right)\gamma^\nu - \gamma^\mu\left(2\eta^{\alpha\nu} - \gamma^\alpha\gamma^\nu\right)\gamma^\beta \\
&= 2\eta^{\beta\mu}\gamma^\alpha\gamma^\nu - \left(2\eta^{\alpha\mu} - \gamma^\mu\gamma^\alpha\right)\gamma^\beta\gamma^\nu + \gamma^\mu\gamma^\alpha\left(2\eta^{\nu\beta} - \gamma^\beta\gamma^\nu\right) - 2\eta^{\alpha\nu}\gamma^\mu\gamma^\beta \\
&= 2\eta^{\beta\mu}\gamma^\alpha\gamma^\nu - 2\eta^{\alpha\mu}\gamma^\beta\gamma^\nu + 2\eta^{\nu\beta}\gamma^\mu\gamma^\alpha - 2\eta^{\alpha\nu}\gamma^\mu\gamma^\beta
\end{aligned}
$$

Question 9

We need a unitary transformation such that

$$\gamma_D^\mu = U\gamma_W^\mu U^\dagger$$

which we can re-write as (given the U is unitary)

$$\gamma_D^\mu U = U\gamma_W^\mu \ .$$

Let $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then for $\mu = 0$:

$$
\begin{aligned}
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\begin{pmatrix} a & b \\ -c & -d \end{pmatrix} &= \begin{pmatrix} b & a \\ d & c \end{pmatrix}
\end{aligned}
\tag{1}
$$

That is, $a = b$ and $c = -d$. Furthermore

$$UU^\dagger = 1 \Rightarrow \begin{pmatrix} a & a \\ c & -c \end{pmatrix}\begin{pmatrix} a^* & c^* \\ a^* & -c^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which further implies that $|a|^2 = \frac{1}{2}$ and $|c|^2 = \frac{1}{2}$. Taking the positive sign implies

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \ .$$

We can check that this is a valid solution by showing that

$$
\begin{aligned}
\gamma_D^i &= U\gamma_W^i U^\dagger \\
\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}
\end{aligned}
$$

Question 10

a) Through a simple application of the Dirac equation, we have

$$s.p = \frac{1}{2m}\bar{u}(p,\lambda)\not{p}\gamma_5 u(p,\lambda) = \frac{1}{2}\bar{u}(p,\lambda)\gamma_5 u(p,\lambda)$$

or, alternatively,

$$s.p = \frac{1}{2m}\bar{u}(p,\lambda)\gamma_5(-\not{p})u(p,\lambda) = -\frac{1}{2}\bar{u}(p,\lambda)\gamma_5 u(p,\lambda) \; .$$

Thus $s.p = 0$.

b) For a particle at rest, where we have $u(p,\lambda) = \sqrt{2m}\begin{pmatrix}1\\0\end{pmatrix}\chi_\lambda$, $p^\mu = (m,0,0,0)$ and $s.p = 0$ we get

$$s_0 = 0$$

and

$$\begin{aligned}\vec{s} &= \frac{1}{2m}\bar{u}(0,\lambda)\gamma_i\gamma_5 u(0,\lambda) \\ &= \chi_\lambda^\dagger(1,0)\begin{pmatrix}0 & \vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\chi_\lambda = \chi_\lambda^\dagger\vec{\sigma}\chi_\lambda \; .\end{aligned}$$

Thus $s_1 = s_2 = 0$ and

$$s_3 = \begin{cases} 1 & \text{for} \quad \chi_+ \\ -1 & \text{for} \quad \chi_- \end{cases} .$$

This means **s** is in the direction of the spin. In this simple frame we have

$$s^\mu = (0,0,0,\pm 1) \; , \quad s^2 = -1 \; .$$

c) The spin vector

$$s_\mu(p,\lambda) = \frac{1}{2m}\bar{u}(p,\lambda)\gamma_\mu\gamma_5 u(p,\lambda)$$

transforms as a four-vector under Lorentz transformations. Thus $s^2 = s^\mu s_\mu$ is a Lorentz scalar and $s^2 = -1$ in all frames.

d) Since $\vec{\eta}$ and $\vec{p}$ are the only vectors in the problem, we can write

$$\vec{s} = a\vec{\eta} + b\vec{p} \; , \quad a \text{ and } b \text{ are constants.}$$

Since we are given $\vec{s} = \vec{\eta}$ when the particle is at rest $\vec{p} = 0$, we mut have $a = 1$. From $s.p = 0$, we get

$$s_0 = \frac{1}{E}(\vec{\eta} + b\vec{p})\cdot\vec{p} = \frac{1}{E}(\vec{\eta}\cdot\vec{p} + bp^2)$$

and the condition $s^2 = -1$ can now be written as

$$s_0^2 - \vec{s}^2 = s_0^2 - (\vec{\eta} + b\vec{p})^2 = -1$$

which leads to

$$\frac{1}{E^2}(\vec{\eta} + b\vec{p})^2 = (\vec{\eta} + b\vec{p})^2 - 1$$

or

$$b^2(E^2 - m^2)m^2 + 2b(\vec{\eta}\cdot\vec{p})m^2 - (\vec{\eta}\cdot\vec{p})^2 = 0$$

or

$$[m(E-m)b + (\vec{\eta}\cdot\vec{p})]\,[m(E-m)b - (\vec{\eta}\cdot\vec{p})] = 0 \; .$$

This gives the solution

$$b = \frac{(\vec{\eta}\cdot\vec{p})}{m(E+m)} \; .$$

(The other solution does not go to zero as $\vec{p} \to 0$.) Thus we have

$$\vec{s} = \vec{\eta} + \frac{\vec{p}(\vec{\eta}\cdot\vec{p})}{m(E+m)} \; .$$