Physics Honours: Standard Model

Tutorial Sheet 6

Question 1

Consider a scalar QED with Higgs phenomena system using the Lagrangian

$$\mathcal{L} = (\mathcal{D}_\mu \phi^\dagger)(\mathcal{D}^\mu \phi) + \frac{\mu^2}{2}\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

with $\mathcal{D}_\mu\phi = (\partial_\mu - ieA_\mu)\,\phi$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Consider the static case where $\partial^0\phi = \partial^0\mathbf{A} = 0$ and $A_0 = 0$. Recall that $\epsilon^{ijk}B^k = -F^{ij}$ and $E^i/c = -F^{0i}$.

a) Show that the equation of motion for $\mathbf{A}$ is of the form

$$\nabla \times \mathbf{B} = \mathbf{J} \quad \text{with} \quad \mathbf{J} = ie\left[\phi^\dagger(\nabla - ie\mathbf{A})\phi - (\nabla + ie\mathbf{A})\phi^\dagger\phi\right] \ .$$

b) Show that with spontaneous symmetry breaking, in the classical approximation $\phi = v = \sqrt{\mu^2/\lambda}$, the current $\mathbf{J}$ is of the form

$$\mathbf{J} = e^2v^2\mathbf{A} \quad \text{(the London equation)}$$

and thus

$$\nabla^2\mathbf{B} = e^2v^2\mathbf{B} \quad \text{(the Meissner effect)} \ .$$

c) The resistivity ρ for the system is defined by

$$\mathbf{E} = \rho\mathbf{J} \ .$$

Show that, in this case of spontaneous symmetry breaking, $\rho = 0$, and we have superconductivity.

Question 2

Consider the Lagrangian of the local $O(3)$ symmetry

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu)_{ij}\phi_j(D^\mu)_{ik}\phi_k - \frac{\mu^2}{2}\phi_i\phi_i - \frac{\lambda}{4}(\phi_i\phi_i)^2$$

with covariant derivative

$$(D_\mu)_{ij} = \delta_{ij}\partial_\mu - ig\,(T_a)_{ij}\,W^a_\mu$$

and $(T_a)_{ij} = -i\epsilon_{aij}$. Choose the solution with spontaneous symmetry breaking ($\mu^2 < 0$) and the vacuum of the theory along the 3 direction:

$$\phi_i = v\,\delta_{i3}$$

and show that both gauge fields W_1^μ and W_2^μ associated with broken generators T_1 and T_2 have weight g^2v^2. Also show that W_3^μ has zero mass.

Question 3

Find the value of the electric charge for the leptons in the doublet Q_L.

Answers

Question 1

a) In the static limit, Maxwell's equations are of the form

$$-\partial_\mu F^{\mu\nu} = J^\nu \Rightarrow \partial_i F^{ij} = -j^j \quad \text{or} \quad \nabla \times \mathbf{B} = \mathbf{J}$$

where

$$\begin{aligned} J^\mu &= -\frac{\partial \mathcal{L}}{\partial A_\mu} = ie\left[(\mathcal{D}_\mu \phi^\dagger)\phi - \phi^\dagger(\mathcal{D}_\mu \phi)\right] \\ &= ie\left[(\partial_\mu + ieA_\mu)\phi^\dagger \phi - \phi^\dagger(\partial_\mu - ieA_\mu)\phi\right] \ . \end{aligned}$$

Thus

$$\mathbf{J} = ie\left[\phi^\dagger(\nabla - ie\mathbf{A})\phi - (\nabla + ie\mathbf{A})\phi^\dagger\phi\right] \ .$$

b) Spontaneous symmetry breaking gives $\phi = v = \sqrt{\mu^2/\lambda}$, which in turn gives the London equation (the terms $\nabla\phi = 0$)

$$\mathbf{J} = e^2 v^2 \mathbf{A} \ .$$

From Maxwell's equation, $\nabla \times \mathbf{B} = \mathbf{J}$, we get

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla \times \mathbf{J} \quad \text{or} \quad \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = -e^2 v^2 \nabla \times \mathbf{A}$$

or

$$\nabla^2 \mathbf{B} = e^2 v^2 \mathbf{B}$$

where we have used $\nabla \cdot \mathbf{B} = 0$. It is not difficult to see that this equation implies the Meissner effect because if implies a solution for the magnetic field of the form

$$B(\mathbf{x}) \simeq \exp\left(\frac{\mathbf{n}\cdot\mathbf{x}}{l}\right) \qquad \text{with} \quad l = \frac{1}{ev} \ .$$

This means that the magnetic field decays in a distance of order $l \sim (ev)^{-1}$.

c) Since $\partial^0 \mathbf{A} = 0$ and $A^0 = 0$, we get $\mathbf{E} = 0$. On the other hand, we have $\mathbf{J} \neq 0$. This means that the resistivity must vanish (superconductivity) $\rho = 0$.

Question 2

The mass term for gauge fields in the Lagrangian is

$$-\frac{1}{2} g^2 v^2 (T_a)_{i3}(T_b)_{i3} W^a_\mu W^{a\mu}$$

and the mass matrix is

$$(M^2_W)_{ab} = -g^2 v^2 (T_a)_{i3}(T_b)_{i3} \ .$$

Using $(T_a)_{ij} = -i\epsilon_{aij}$ we obtain

$$(T_a)_{i3}(T_b)_{i3} = -\epsilon_{ai3}\epsilon_{bi3} = -(\delta_{ab} - \delta_{a3}\delta_{b3})$$

and the explicit form of the mass matrix:

$$(M^2_W)_{ab} = g^2 v^2 (\delta_{ab} - \delta_{a3}\delta_{b3}) = g^2 v^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So the two fields W^μ_1 and W^μ_2 associated with broken generators T_1 and T_2 have weight $g^2 v^2$ whereas W^μ_3 has zero mass since it is associated with the unbroken $O(2)$ symmetry.

Question 3

Using the formula

$$Q_{\text{em}} = I_{3w} + \frac{1}{2}y \ ,$$

we find

$$Q_{\text{em}} \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L = \left[\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} + \frac{-1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L = \begin{pmatrix} 0 \\ -e \end{pmatrix}_L \ .$$

So the neutrino has no electric charge and the electron has a charge -1.