Physics Honours: Standard Model

Tutorial Sheet 5

Question 1

Calculate the cross-section of the process $\bar{\nu}\, e \to \bar{\nu}\, e$ and check that if we neglect the electron mass $s \gg m^2$:

$$\sigma(\bar{\nu}\, e \to \bar{\nu}\, e) = \frac{1}{3}\sigma(\nu\, e \to \nu\, e)\ .$$

What is the reason for the factor 1/3 between the two cross-sections?

Question 2

Calculate the matrices T_i from infinitesimal $O(3)$ rotations:

$$\phi_j \to (\delta_{jk} + \epsilon_{ijk}\theta\, n_i)\, \phi_k = (1 + i\theta\, T_i\, n_i)_{jk}\, \phi_k\ .$$

Question 3

The form of an $SU(2)$ element in the adjoint representation (eg, a pion triplet) is given by

$$e^{-i\vec{\theta}\cdot\vec{L}} \quad \text{where} \quad (L_i)_{jk} = -i\epsilon_{ijk} \;\; i,j,k = 1,2,3$$

These should be 3×3 rotation matrices. Show explicitly that this is the case for a rotation about the $\hat{z}$ axis.

Question 4

Consider a complex scalar field ϕ_i in the vector representation of $SU(n)$, which transforms as follows under infinitesimal transformations of $SU(n)$

$$\begin{aligned} \phi_i &\to \phi_i + i\epsilon_i^j \phi_j \\ \phi^i &\to \phi^i - i\epsilon_k^i \phi^k \end{aligned}$$

with $\phi_i^* = \phi^i$. Find an expression which is invariant under $SU(n)$ transformations and construct a renormalisable scalar potential for a general theory in 4-dimensions.

Choose a value for the vacuum of the scalar field as

$$\langle 0|\phi|0\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ v \end{pmatrix}$$

and consider the translation of this minimum of the field to study the properties of the components of the scalar field. How many Goldstone bosons remain massless in the spectrum of the theory? What is the residual group invariance of the theory?

Doing the same exercise with two complex scalar fields ϕ_{1i} and ϕ_{2i} in the vector representation of $SU(n)$, where they transform in the same way that ϕ_i previously did. Build the scalar potential and do not forget to also consider the terms which mix the two fields.

Select vacuum expectation values

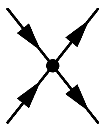
$$\langle 0|\phi_1|0\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ v_1 \end{pmatrix} \qquad\qquad \langle 0|\phi_2|0\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ v_2 \\ v_3 \end{pmatrix}$$

and study the symmetry breaking.

Answers

Question 1

The matrix element is calculated in the following way



$$iM = -i\frac{G_F}{\sqrt{2}}\,[\bar{v}(p')\gamma^{\mu}(1-\gamma_5)u(k)]\,[\bar{u}(k')\gamma_{\mu}(1-\gamma_5)v(p)]$$

where the result is obtained similarly to that of $\nu_e\, e \to \nu_e\, e$. The differential cross-section is:

$$\frac{d\sigma}{d\Omega} = \frac{G_F^2}{4\pi^2}\frac{(u-m^2)^2}{s}\ .$$

To obtain the total cross-section simply integrate over the angles, $u \simeq -s(1-x)$:

$$\sigma(\bar{\nu}\, e \to \bar{\nu}\, e) = \frac{G_F^2}{3\pi}(s-m^2)\left(1-\frac{m^6}{s^3}\right)\ .$$

In the limit $s \gg m^2$

$$\sigma(\bar{\nu}\, e \to \bar{\nu}\, e) \simeq \frac{G_F^2}{3\pi}\ s\ .$$

The factor 1/3 is due to the neutrino helicity. Suppose we take the z-axis along the direction of the incident particles (for example, with a positive sign in the direction of the momentum of the incoming electron). In $\nu_e\, e \to \nu_e\, e$, the initial state is in a spin state $J_z = 0$ because both incoming particles have a left-handed helicity (and momentum in an opposite direction). There is no restriction on the direction of the outgoing particles from the elastic collision, where in terms of conservation of total spin.

$$\text{entering}: \quad J_z = 0 \qquad \text{exiting }(\theta=\pi): \quad J_z = 0$$

$$e \overset{\Leftarrow}{\longrightarrow}\overset{\Rightarrow}{\longleftarrow} \nu_e \qquad\qquad e \overset{\Rightarrow}{\longleftarrow}\overset{\Leftarrow}{\longrightarrow} \nu_e$$

In particular the incoming particles can bounce straight back after the collision ($\theta = \pi$). However, in $\bar{\nu}_e\, e \to \bar{\nu}_e\, e$ the anti–neutrino is a particle of right-handed helicity

$$\text{entering}: \quad J_z = -1 \qquad \text{exiting }(\theta=\pi): \quad J_z = +1$$

$$e \overset{\Leftarrow}{\longrightarrow}\overset{\Leftarrow}{\longleftarrow} \bar{\nu}_e \qquad\qquad e \overset{\Rightarrow}{\longleftarrow}\overset{\Rightarrow}{\longrightarrow} \bar{\nu}_e$$

The total spin of the initial system is $J_z = -1$ and for $\theta = \pi$ the final state of total spin $J_z = +1$ is forbidden by conservation of angular momentum.

Question 2

It suffices to note that

$$(T_i)_{jk} = -i\epsilon_{ijk}$$

and to write the matrices

$$T_1 = -i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\ ,\ T_2 = -i\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\ ,\ T_3 = -i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\ .$$

Question 3

$(L_3)_{jk} = -i\epsilon_{3jk}$ where the only non-vanishing entries are $\epsilon_{312} = -\epsilon_{321} = 1$

$$\Rightarrow L_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{Note}: \; L_3^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} \text{Therefore } e^{-i\theta_3 L_3} &= \sum_{n\ even} \frac{(-i\theta_3)^2}{n} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \sum_{n\ odd} \frac{(-i\theta_3)^2}{n} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta_3 & 0 & 0 \\ 0 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -i(-i)\sin\theta_3 & 0 \\ (-i)i\sin\theta_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Question 4

An expression invariant under the transformations of $SU(n)$ is given by the scalar product in the complex vector space

$$\phi_i\phi^i \rightarrow (\phi_i + i\epsilon_i^j\phi_j)\,(\phi^i - i\epsilon_k^i\phi^k) = \phi_i\phi^i \; .$$

The renormalisable invariant potential can be constructed from this invariant combination

$$V(\phi) = \mu^2\phi_i\phi^i + \frac{\lambda}{2}\left(\phi_i\phi^i\right)^2 \; .$$

For $\mu^2 < 0$ the minimum potential is given by

$$\phi_i\phi^i = \sqrt{\frac{-\mu^2}{\lambda}} \equiv v \; .$$

The value in the vacuum for the scalar field is chosen in the direction n of the potential

$$\langle 0|\phi_i|0\rangle = \delta_{in}\, v$$

as indicated by the exercise. The symmetry is broken as follows

$$SU(n) \rightarrow SU(n-1) \; .$$

The number of Goldstone bosons is given by the number of generators broken by the theory which is in turn given by the difference between the number of generators of $SU(n)$, $n^2 - 1$ and $SU(n-1)$, $[(n-1)^2 - 1]$;

$$(n^2 - 1) - [(n-1)^2 - 1] = 2n - 1 \; .$$

To study in more detail the symmetry breaking we can place at least one translation field

$$\phi_i = +\delta_{in}\, v$$

in the theory and write the potential in terms of the new fields. The quadratic part of the potential gives mass terms

$$\mu^2\,(\phi_i'\phi'^i) + \frac{\lambda}{2}\left[v^2(\phi_n + \phi^n)^2 + 2v^2(\phi_i'\phi'^i)\right] = -\frac{\mu^2}{2}(\phi_n + \phi^n)^2 \; .$$

The fields ϕ_i are complex (two degrees of freedom for each field) and $\phi^i = \phi_i^*$. Only the real part of ϕ_n has mass. The other $2n - 1$ fields are the massless Goldstone bosons.

With two multiplets of scalar complex fields we have four invariant combinations

$$\phi_{1i}\phi^{1i} \; , \qquad \phi_{2i}\phi^{2i} \; , \qquad \phi_{1i}\phi^{2i} \; , \qquad \phi_{2i}\phi^{1i}$$

from which to build the invariant potential. The pattern of symmetry breaking is as follows

$$SU(n) \to SU(n-2)$$

with

$$(n^2-1) - [(n-2)^2 - 1] = 4n - 4$$

Goldstone bosons.