# Introduction to the Standard Model

**presented by Alan S. Cornell**
National Institute for Theoretical Physics (NITheP);
School of Physics, University of the Witwatersrand

NITheP
National Institute for Theoretical Physics

# Particle content of the SM

| Fermions | $1^{st}$ generation | | $2^{nd}$ generation | | $3^{rd}$ generation | |
|---|---|---|---|---|---|---|
| $\mathrm{Q}\left\{\begin{array}{c}\mathrm{U}\\ \mathrm{D}\end{array}\right.$ | $\begin{pmatrix} u \\ d \end{pmatrix}_L$ | $\begin{array}{c} u_R \\ d_R \end{array}$ | $\begin{pmatrix} c \\ s \end{pmatrix}_L$ | $\begin{array}{c} c_R \\ s_R \end{array}$ | $\begin{pmatrix} t \\ b \end{pmatrix}_L$ | $\begin{array}{c} t_R \\ b_R \end{array}$ |
| $\mathrm{L}\left\{\begin{array}{c}\mathrm{E}\\ \mathrm{N}\end{array}\right.$ | $\begin{pmatrix} e \\ \nu_e \end{pmatrix}_L$ | $e_R$ | $\begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix}_L$ | $\mu_R$ | $\begin{pmatrix} \tau \\ \nu_\tau \end{pmatrix}_L$ | $\tau_R$ |

| Gauge bosons | $\gamma$ | $W^\pm$ , $Z$ | $G^a$ |
|---|---|---|---|

plus the Higgs boson $H$.

The Lagrangian of the Standard Model is given by

$$\mathcal{L}_{\mathrm{SM}} = \mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{D}} + \mathcal{L}_{\mathrm{Yukawa}} + \mathcal{L}_{\mathrm{Higgs}}$$

The kinetic part
of the gauge fields

The Yukawa sector
(interactions between the
Higgs doublet and fermions)

The Dirac fermions

Higgs dynamics and EWSB

In this course we shall not consider possible gauge-fixing and ghost field contributions (which may result from other choices of gauge)

We shall construct our Lagrangian using the following rules:

1) The action should be real

2) $\mathcal{L}$ should be bilinear in the fields (for free fields)

3) $\mathcal{L}$ should be invariant under the symmetries we desire

# Spinors and Lorentz transformations

So with this is mind, let us construct a representation of fermions that transform under rotations (a subgroup of the Lorentz group)

Recall that fermions $|\psi\rangle = \begin{pmatrix} |\psi_\uparrow\rangle \\ |\psi_\downarrow\rangle \end{pmatrix}$ have two components (say spin-up and spin-down), and spin operators $S_i = \frac{1}{2}\sigma_i$ for Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad and \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

So if we consider a case where we have orbital angular momentum (total angular momentum) **J** given by the spin-operators,

$$|\psi\rangle \to U_R(\hat{e},\theta)|\psi\rangle$$

where $U_R(\hat{e},\theta) = \exp(-i\mathbf{J}\cdot\hat{e}\theta) = \exp(-i\vec{\sigma}\cdot\hat{e}\theta/2)$

Note that we want the transformations properties of the full Lorentz group, not just rotations. So I shall introduce a *simple* trick for obtaining the spinor representation from the vector representation, which is **not** generalisable to other representations.

Recall that representations are the group of transformations which take $(x, y, z) \rightarrow (x', y', z')$ whilst leaving $r^2 = x^2 + y^2 + z^2$ invariant.

The Lorentz group of transformations takes $(t, x, y, z) \rightarrow (t', x', y', z')$ leaving $t^2 - x^2 - y^2 - z^2$ invariant.

So to start let us assemble the components of a 3-vector into a $2 \times 2$ traceless Hermitian matrix

$$X = \left( \begin{array}{cc} z & x - iy \\ x + iy & -z \end{array} \right) = x_i \sigma_i$$

Consider the transformation $X' = UXU^\dagger$, where $U$ is a $2 \times 2$ unitary matrix with unit determinant $\Rightarrow \mathrm{Tr} X' = \mathrm{Tr}\ X$, $X'^\dagger = X'$ and

$$X' = \begin{pmatrix} z' & x' - iy' \\ x' + iy' & -z' \end{pmatrix}$$

Since $\det X' = \det X \Rightarrow x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2$.
So this is another way to write the transformation law of vectors under rotations, where $U = \exp(-i\vec{\sigma} \cdot \hat{e}\theta/2)$

As the $U$'s by themselves form a two-dimensional representation of the rotation group, and a spinor is a two-component column vector, it transforms under rotations as $u' = Uu$.

Generalising to the full Lorentz group by writing

$$X = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$$

consider the transformation $X' = QXQ^\dagger$, where $Q$ is no longer required to be unitary, in general, but $\det Q = 1$. Then

$$\begin{aligned} \det X' &= \det X \\ \Rightarrow t'^2 - x'^2 - y'^2 - z'^2 &= t^2 - x^2 - y^2 - z^2 \end{aligned}$$

and so the transformation corresponds to a Lorentz transformation. Where a spinor transforms as $u \to Qu$.

However, this construction is not unique, and we have two representations of the Lorentz group for spinors which are **inequivalent**. Therefore we have two different types of spinor fields, one that transforms according to $Q$, and one for $Q^*$.

Note that for the rotations subgroup $U$ and $U^*$ are equivalent representations (we can use $i\sigma_2$ to switch between the two, that is, they are related by a change of basis)

$$U(R) = \exp(-i\vec{\sigma} \cdot \hat{e}\theta/2) = i\sigma_2 U^*(R)(i\sigma_2)^\dagger$$

However, for boosts

$$i\sigma_2(\exp(-\vec{\sigma}\cdot\hat{e}\phi/2))^*(i\sigma_2)^\dagger = \exp(\vec{\sigma}\cdot\hat{e}\phi/2) \neq \exp(-\vec{\sigma}\cdot\hat{e}\phi/2)$$

Thus we can define two types of spinors $u_\pm$ which transform under rotations as

$$u_\pm \to \exp(-i\vec{\sigma}\cdot\hat{e}\theta/2)u_\pm$$

but differently under boosts

$$u_\pm \to \exp(\pm\vec{\sigma}\cdot\hat{e}\phi/2)u_\pm$$

Let us move on to the question of how to construct Lorentz invariant Lagrangians which are bilinear in the fields. As such we shall now need to know how terms bilinear in the u's transform

Since in some sense spinors were square roots of the vectors, we can construct four-vectors from pairs of spinors.

First consider $u_+^\dagger u_+ \to u_+^\dagger Q^\dagger Q u_+$ under a Lorentz transformation (similarly for $u_-^\dagger u_-$). For rotations $Q$ is unitary, therefore $u_+^\dagger u_+$ is a scalar under rotations, but not so for boosts!

For $u_+^\dagger \vec{\sigma} u_+$ they form a three-vector under rotations (similarly $u_-^\dagger \vec{\sigma} u_-$).

Putting these together as $V^\mu = (u_+^\dagger u_+, u_+^\dagger \vec{\sigma} u_+)$ and $W^\mu = (u_-^\dagger u_-, u_-^\dagger \vec{\sigma} u_-)$, these transform as four-vectors under a proper Lorentz transformation.

# The Weyl Lagrangian

We can now construct a Lagrangian for $u_+$, keeping in mind the following restrictions:

1) The action should be real

2) $\mathcal{L}$ should be bilinear in the fields

3) $\mathcal{L}$ should be invariant under the internal symmetry

$$u_+ \to e^{-i\lambda} u_+ \ , \ u_+^\dagger \to e^{i\lambda} u_+$$

as all fermions in the real world carry some conserved charge (like baryon number or lepton number)

We've already seen how to construct invariant four-vectors from $u_+$ and $u_+^\dagger$, to make this a scalar we just contract it with another vector.

The only other vector we have at our disposal is $\partial_\mu$.

Hence the simplest Lagrangian we can write down that satisfies our requirements is

$$\mathcal{L} = i\left(u_+^\dagger \partial_0 u_+ + u_+^\dagger \vec{\sigma} \cdot \vec{\nabla} u_+\right)$$

the $i$ in front is to make the action real, and the sign of $\mathcal{L}$ is not fixed at this point, where we shall take it to be positive.

Note that the positivity of the energy will be a problem regardless of our choice of sign.

# The Dirac equation

However, we want a theory for massive fermions, like the electron, which conserves parity (as EM and strong interactions conserve parity)

Now as the parity operator $P : u_\pm(\vec{x}, t) \to u_\mp(-\vec{x}, t)$, a parity invariant theory must have both types of spinors

The simplest Lagrangian is just

$$\mathcal{L}_0 = iu_+^\dagger \left(\partial_0 + \vec{\sigma} \cdot \vec{\nabla}\right) u_+ + iu_-^\dagger \left(\partial_0 - \vec{\sigma} \cdot \vec{\nabla}\right) u_-$$

but this is nothing more than two decoupled massless spinors

However, as $u_+^\dagger u_-$ and $u_-^\dagger u_+$ transform as scalars under a Lorentz transformation, we can add them

$$\mathcal{L} = \mathcal{L}_0 - m(u_+^\dagger u_- + u_-^\dagger u_+)$$

and this new term has dimension of mass

This Lagrangian leads to the coupled equations of motion

$$\begin{aligned} i\left(\partial_0 + \vec{\sigma}\cdot\vec{\nabla}\right)u_+ &= mu_- \\ i\left(\partial_0 - \vec{\sigma}\cdot\vec{\nabla}\right)u_- &= mu_+ \end{aligned}$$

multiplying by $\partial_0 \pm \vec{\sigma} \cdot \vec{\nabla}$

$$\Rightarrow (\partial^\mu \partial_\mu + m^2) u_\pm = 0$$

the massive Klein-Gordon equation

If we now group the two fields as a single four-component *bi-spinor* $\psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$, we get the Dirac equation in the **Weyl representation**, that is

$$\mathcal{L} \to i\psi^\dagger \partial_0 \psi + i\psi^\dagger \vec{\alpha} \cdot \vec{\nabla} \psi - m\psi^\dagger \beta \psi$$

where $\vec{\alpha} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}$ , $\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\Rightarrow$ equations of motion $i(\partial_0 + \vec{\alpha} \cdot \vec{\nabla})\psi = \beta m \psi$
(**The Dirac equation**).

In terms of $\psi$,
Parity transformation are $P : \psi(\vec{x}, t) \rightarrow \beta\psi(-\vec{x}, t)$
and a Lorentz boost $\psi \rightarrow \exp(\vec{\alpha} \cdot \hat{e}\phi/2)\psi$
Since $u_+$ and $u_-$ transform the same way under rotations,
$\psi$ transforms as $R : \psi \rightarrow \exp(\vec{L} \cdot \hat{e}\theta/2)\psi$ for $\vec{L} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}$
with $\{\beta, \alpha_i\} = 0$ , $\{\alpha_i, \alpha_j\} = 2\delta_{ij}$ and $\beta^2 = 1$.

Note that this representation is not unique, we could have defined $\psi = \frac{1}{\sqrt{2}}\begin{pmatrix} u_+ + u_- \\ u_+ - u_- \end{pmatrix}$

$$\Rightarrow \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \; \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which will give all the same commutation relations for $\alpha$ and $\beta$, this being the **Dirac representation** you may have seen elsewhere.

There are other ways we can represent fermions in terms of chirality, or even Majorana particles etc.

# $\gamma$ matrices

To make our theory for fermions look Lorentz invariant, we define the Dirac adjoint for a bispinor $\psi$ as

$$\bar{\psi} \equiv \psi^\dagger \beta$$

Therefore $\bar{\psi}_1 \psi_2$ is a scalar under a Lorentz transformation

Furthermore, we know that $(\psi_1^\dagger \psi_2, \psi_1^\dagger \vec{\alpha} \psi_2) = (\bar{\psi}_1 \beta \psi_2, \bar{\psi}_1 \beta \vec{\alpha} \psi_2)$ transform like a four-vector, in which case we define

$$\gamma^0 \equiv \beta \text{ and } \gamma^i \equiv \beta \alpha_i$$

# Chirality and $\gamma_5$

We shall now work in the Weyl basis, where
$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$.

In which case we define projection operators

$$P_R = \frac{1}{2}(1+\gamma_5) \quad P_L = \frac{1}{2}(1-\gamma_5)$$

(this definition applies in any basis, not just the Weyl) and that these operators project out our Weyl spinors, that is

$$
\begin{aligned}
\begin{pmatrix} u_+ \\ 0 \end{pmatrix} &= \frac{1}{2}(1+\gamma_5)\psi &= \quad & P_R\psi \equiv \psi_R \\
\begin{pmatrix} 0 \\ u_- \end{pmatrix} &= \frac{1}{2}(1-\gamma_5)\psi &= \quad & P_L\psi \equiv \psi_L
\end{aligned}
$$

where $\psi_L$ and $\psi_R$ are just the left and right-handed pieces of the Dirac spinor in four-component form, rather than the two-component notation earlier.

The Dirac Lagrangian is then

$$
\mathcal{L} = \underbrace{\bar{\psi}_L i\gamma^\mu\partial_\mu\psi_L + \bar{\psi}_R i\gamma^\mu\partial_\mu\psi_R}_{\text{two independent helicity state equations which are}} - \underbrace{m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)}_{\text{coupled (left and right-handed fields) by the mass term}}
$$

# Local Gauge Invariance

Let us consider transformations which do depend on the space-time coordinates ($\theta^a = \theta^a(x)$). One speaks in this case of local or gauge symmetries (Weyl 1929).

The advantage of gauge symmetries is that from a free theory invariant under global transformations it is possible to construct a theory invariant under local transformations (gauge transformations) by adding interaction terms and one or more vector fields (gauge fields).

How to introduce these terms is not arbitrary but the imposition of the invariance of the Lagrangian under gauge transformations allows us to "generate" interactions and introduce vector fields which are the mediators of forces.

# Example: Electromagnetism

Consider, as a starting point, the Dirac equation for a free electron

$$\mathcal{L}_D = \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x)$$

which is invariant under global $U(1)$ transformations:

$$\begin{aligned} \psi(x) \to \psi'(x) &= e^{-i\alpha}\psi(x) \\ \bar{\psi}(x) \to \bar{\psi}'(x) &= e^{i\alpha}\bar{\psi}(x) \, . \end{aligned}$$

The corresponding local symmetry is:

$$\begin{aligned} \psi(x) \to \psi'(x) &= e^{-i\alpha(x)}\psi(x) \\ \bar{\psi}(x) \to \bar{\psi}'(x) &= e^{i\alpha(x)}\bar{\psi}(x) \, . \end{aligned}$$

The mass term of the Lagrangian is invariant under the local transformation, but the derivative term is not:

$$\bar{\psi}(x)\partial_\mu\psi(x) \to \bar{\psi}(x)\partial_\mu\psi(x) - i\bar{\psi}(x)[\partial_\mu\alpha(x)]\psi(x) \; .$$

To offset this additional term one can define a covariant derivative with the property:

$$D_\mu\psi(x) \to e^{-i\alpha(x)}D_\mu\psi(x)$$

which provides an invariant term in the Lagrangian $\bar{\psi}(x)D_\mu\psi(x)$.

The covariant derivative is obtained with the introduction of a vector field (gauge field) $a_\mu(x)$:

$$D_\mu \psi(x) = (\partial_\mu + iea_\mu)\psi(x)$$

where the gauge field transforms under $U(1)$ by:

$$a_\mu(x) \to a'_\mu(x) = a_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x) \ .$$

The gauge field is not currently a dynamic field, it can be eliminated using the equations of motion. To make it physical we must add a kinetic term.

A term which is gauge invariant and derived from the field $a_\mu$ (which is also renormalisable), such as $f_{\mu\nu}(x)f^{\mu\nu}(x)$, where

$$f_{\mu\nu}(x) = \partial_\mu a_\nu(x) - \partial_\nu a_\mu(x) \ .$$

With the usual normalisation for the kinetic term, the Lagrangian deduced from Dirac's Lagrangian with the application of local invariance is

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(x)(i\gamma^\mu D_\mu - m)\psi(x) - \frac{1}{4} f_{\mu\nu}(x) f^{\mu\nu}(x) \ ,$$

which is the Lagrangian of quantum electrodynamics (QED).

It may be noted that there is an absence of a mass term for the field $a_\mu$. The reason for this is that the mass term $m a_\mu a^\mu$ is not gauge invariant. The photon is therefore massless.

## Example: The Yang-Mills theory

Electromagnetism can be generalised (Yang and Mills 1954) to rotation by a phase, where the phase is a matrix:

$$\psi \to S\psi$$

with $S$ being a special unitary matrix, for example $S \in SU(2)$ and $\psi$ a doublet.

The invariance of physics in relation to local rotations of $SU(2)$

$$S(x) = e^{-i\theta^a(x)\sigma^a/2} \qquad a = 1, 2, 3$$

the $\sigma^a$ being the Pauli matrices, can be done in analogy if we consider an infinitesimal transformation of $SU(2)$

$$S(x) \simeq 1 - i\frac{\theta^a(x)\sigma^a}{2} \ ,$$

where the transformation of the vector field $A_\mu(x)$ is

$$A^i_\mu(x) \to A^i_\mu(x) - \frac{1}{g}\partial_\mu\theta^i + \epsilon^{ijk}\theta^j(x)A^k_\mu(x) \ .$$

Compared to the Abelian case we have an $\epsilon^{ijk}$ term and $A^i_\mu$ transforms as a triplet in the adjoint representation of $SU(2)$. So the fields $A^i_\mu$ are charged against the charge of $SU(2)$ whilst for $U(1)$ we had a neutral field (the photon) compared to the charge of $U(1)$ (electric charge). The tensor $F_{\mu\nu}(x)$:

$$F^i_{\mu\nu}(x) = \partial_\nu A^i_\mu(x) - \partial_\mu A^i_\nu(x) + g\epsilon^{ijk} A^j_\mu(x) A^k_\nu(x)$$

is a triplet under the gauge transformation of $SU(2)$:

$$F^i_{\mu\nu}(x) \to F^i_{\mu\nu}(x) + \epsilon^{ijk}\theta^j(x) F^k_{\mu\nu} \ .$$

The tensor $F^i_{\mu\nu}(x)$ is not gauge invariant, however, the product

$$Tr\left[(\sigma^a F^a_{\mu\nu}(x))(\sigma^b F^{b\mu\nu}(x))\right] \propto F^i_{\mu\nu}(x) F^{i\mu\nu}(x) \ ,$$

which we will use in the Lagrangian, is invariant. In terms of some of the other differences with the Abelian theory is the presence of self-interaction terms for the gauge fields in the kinetic term.

# Symmetry breaking

Before progressing with a closer analysis of this Lagrangian and its components, we first need some background theory. We shall start with a look at symmetries, where in quantum mechanics an exact (unbroken) symmetry $T$ has the property of transforming the states of a system:

$$T : \phi \to \phi'$$

such that the transition probabilities do not change

$$|\langle \phi, \psi \rangle|^2 = |\langle \phi', \psi' \rangle|^2 \ .$$

The operator $U$ of the transformation is unitary or anti-unitary, and in terms of an observable $A$

$$T : A \to A' = U A U^{-1} \ .$$

Such a transformation preserves the commutation relations and more general algebraic relations, especially any equations of motion in the theory do not change under the transformation $T$.

Conversely one may ask whether a symmetry of the equations of motion implies an exact symmetry. The answer is yes for a system with a finite number of degrees of freedom.

If the number of degrees of freedom of the theory is infinite (as in field theory) the answer is no. The reason is the presence of nonequivalent representations of canonical commutation relations. The symmetry of the equations of motion may not give rise to transformations of system states which preserve the transition probability. One speaks in this case of spontaneously broken symmetries.

# Spontaneously broken continuous symmetries

As we build up to the Goldstone theorem, consider a scalar theory with an $O(3)$ symmetry

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4 \ .$$

The notation is compact, $\phi \equiv (\phi_1, \phi_2, \phi_3)$ is a vector of $O(3)$ and $\phi^2$ is the scalar product $\phi \cdot \phi$, the fourth power of $\phi$ means $\phi^4 = (\phi \cdot \phi)^2$.

An infinitesimal rotation through the angle $\theta$ in the direction of the vector $n$ (with $|n|^2 = 1$) can be written as

$$\phi \to \phi + \theta\,\phi \wedge n \ .$$

Since a rotation leaves the length of a vector invariant, for an infinitesimal rotation we can write

$$|\phi|^2 \to |\phi + \delta\phi|^2 = |\phi|^2 + 2\phi \cdot \delta\phi + \mathcal{O}(\delta\phi^2)$$

and conclude that $\phi$ and $\delta\phi$ are orthogonal

$$\phi \cdot \delta\phi = 0$$

in order to keep the vector invariant. By definition of the vector product a rotation around the direction $n$, $\delta\phi$ must also be orthogonal to $n$, as follows by comparing the above formulas

$$\delta\phi = \theta \; \phi \wedge n \; .$$

For example if $n \equiv (0,\, 0,\, 1)$ one finds

$$\delta\phi_1 = \theta\ \phi_2 \qquad\qquad \delta\phi_2 = -\theta\ \phi_1 \qquad\qquad \delta\phi_3 = 0\ .$$

The minimum of the potential is given by

$$\frac{\partial V}{\partial \phi_i} = \mu^2 \phi_i + \lambda \phi_i |\phi|^2 = 0$$

with two possible solutions

$$\phi_i = 0, \qquad\quad \text{or} \qquad\quad |\phi|^2 = v^2$$

with $v = \sqrt{\dfrac{-\mu^2}{\lambda}}$.

The minimum is found by examining the second derivative

$$\frac{\partial^2 V}{\partial\phi_i \partial\phi_j} = \delta_{ij}(\mu^2 + \lambda|\phi|^2) + 2\lambda\phi_i\phi_j \ .$$

According to the sign of $\mu^2$ we have the following two possibilities:

$$\begin{array}{lllll} \mu^2 & > & 0 & \qquad\qquad & \phi = 0 \ , \\ \mu^2 & < & 0 & \qquad\qquad & |\phi|^2 = v^2 \ . \end{array}$$

If $\mu^2 > 0$ we have a single real minimum $\phi = 0$.

In the case $\mu^2 < 0$ we have an infinite number of degenerate minima, the points on the sphere $|\phi|^2 = v^2$.

By choosing one of these points, for example $\phi_i = \delta_{i3} v$, we can be develop an expansion around the minimum

$$V(\phi) = V|_{min} + \frac{1}{2} \frac{\partial^2 V}{\partial \phi_i \partial \phi_j}\bigg|_{min} (\phi_i - \delta_{i3}\, v)\,(\phi_j - \delta_{j3}\, v)$$

and use the differences $(\phi_i - \delta_{i3} v)$ as new fields to be treated as physical around that minimum. The previous formula indicates the mass of the field after breaking the $O(3)$ symmetry:

$$M^2_{ij} = \frac{\partial^2 V}{\partial \phi_i \partial \phi_j}\bigg|_{min} = -2\mu^2 \delta_{i3}\delta_{j3} = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2\mu^2 \end{array} \right) .$$

So the masses of the fields $\phi_1$ and $\phi_2$ are zero, by the conservation properties of the field $\chi = \phi_3 - v$ is nonzero:

$$m^2_{\phi_1} = m^2_{\phi_2} = 0 \, , \qquad\qquad m^2_\chi = -2\mu^2 \, .$$

The potential in terms of the new fields shows explicitly how the $O(3)$ symmetry is broken:

$$\begin{aligned} V \quad &= \quad -\frac{m_\chi^4}{16\lambda} + \frac{1}{2} m_\chi^2 \chi^2 + \sqrt{\frac{m_\chi^2 \lambda}{2}} \left(\phi_1^2 + \phi_2^2 + \chi^2\right) \chi \\ &\qquad + \frac{\lambda}{4} \left(\phi_1^2 + \phi_2^2 + \chi^2\right)^2 \, . \end{aligned}$$

It may be noted that the Lagrangian has a residual $O(2)$ symmetry, because $V$ depends only on the combination $\phi_1^2 + \phi_2^2$ which is invariant for rotations around the axis $(0,\ 0,\ v)$.

This potential is not usually possible with the $O(2)$ symmetry, the spontaneous breaking of the $O(3)$ symmetry imposes constraints on the shape of the Lagrangian. It was also shown that we obtained a theory with two scalar bosons without mass corresponding to the symmetry breaking by the two axes 1 and 2.

# The Goldstone theorem

In general, if a group with an internal symmetry $G$ is broken spontaneously into a group $H \subset G$ which corresponds to a symmetry of the vacuum state, the number of Goldstone bosons is the number of generators of $G$ minus the number of generators of $H$. Since the size of a group is given by number of generators we can write the number of Goldstone bosons as

$$\dim(G) - \dim(H) = \dim(G/H) ,$$

where $G/H$ is called the quotient group. The physical origin of these massless particles is due to the fact that broken generators allow transitions between degenerate vacuum states (which have the same energy) and these transitions do not cost any energy to the system.

# The Higgs mechanism

The Goldstone theorem is a problem rather than a solution for generating masses. In the spontaneous breaking of a symmetry we obtained massless particles. When one spontaneously breaks a gauge theory the results are very different. The reason is that the Goldstone theorem does not apply to a gauge symmetry because it is impossible to quantify a gauge theory, keeping at the same time the covariance of the theory and that the norm of the Hilbert space remain positive.

In the case of a spontaneously broken gauge theory the gauge bosons corresponding to the broken symmetries have mass and the corresponding Goldstone bosons disappear. We call this phenomenon the Higgs mechanism.

# Example: $O(2)$

We can consider the example of a theory with an $O(2)$ symmetry

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4 \;,$$

with vector fields $\phi$ having two real components. The symmetry $O(2)$ is not a gauge symmetry and we can repeat the analysis of the previous section. If $\mu^2 < 0$ we can choose the vacuum

$$\phi = (v, 0) \;, \qquad v = \sqrt{-\frac{\mu^2}{\lambda}}$$

and make a translation of the field $\phi_1$ to $\phi_1 = \chi + v$, with the potential becoming

$$V = -\frac{m_\chi^4}{16\lambda} + \frac{1}{2} m_\chi^2 \chi^2 + \sqrt{\frac{m_\chi^2 \lambda}{2}} \left(\phi_2^2 + \chi^2\right) \chi + \frac{\lambda}{4} \left(\phi_2^2 + \chi^2\right)^2 .$$

The Goldstone boson $\phi_2$ remains massless and the continuous symmetry, $O(2)$, is completely broken (except for a discrete symmetry $\phi_2 \to -\phi_2$). The infinitesimal transformation under $O(2)$ of the field $\phi$ is given by

$$\delta\phi_1 = -\alpha\phi_2 , \qquad \delta\phi_2 = \alpha\phi_1$$

and in terms of the new fields

$$\delta\chi = -\alpha\phi_2\,, \qquad\qquad \delta\phi_2 = \alpha\chi + \alpha v\ .$$

Thus the Goldstone boson, in terms of new variables, becomes a rotation plus a translation. The invariance of the field to translation makes the potential $V$ flat in this direction, and this in turn means that the translation does not cost any energy and the particle is massless.

We will now analyze the same model in the case of a local symmetry (gauge symmetry). The invariance under transformations of the Goldstone boson becomes

$$\delta\phi_2(x) = \alpha(x)\chi(x) + \alpha(x)v$$

and since $\alpha(x)$ is an arbitrary function of space-time there can be a choice of how to eliminate $\phi_2$. To see these details we can transform to polar coordinates

$$\rho = \sqrt{\phi_1^2 + \phi_2^2}\,, \qquad \theta = \arcsin\frac{\phi_2}{\rho}\,,$$

where the transformation under finite rotations is

$$\rho \to \rho\,, \qquad\qquad \theta \to \theta + \alpha\ .$$

In the case of an infinitesimal systems of two coordinates coinciding:

$$\rho = \sqrt{\phi_2^2 + \chi^2 + 2v\chi + v^2} \sim v + \chi\,, \qquad\qquad \theta \sim \frac{\phi_2}{\chi + v} \sim \frac{\phi_2}{v}\ .$$

we make the theory invariant under local transformations

$$\theta(x) \to \theta(x) + \alpha(x)$$

with the choice $\alpha(x) = -\theta(x)$ and the field $\theta(x)$ can be completely eliminated from the theory.

To explicitly construct the locally invariant theory we must introduce a gauge field and covariant derivatives. It is easier to do this change by writing the real scalar fields in terms of a complex field

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \ , \qquad \phi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$$

and rotations of $O(2)$ become phase transformations for the complex field $\phi$

$$\phi \to e^{i\alpha}\phi \ .$$

The Lagrangian model is written in the new variables

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - \mu^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2 \ .$$

To make the Lagrangian invariant under the local transformations we must introduce covariant derivatives

$$\partial_\mu \phi \to (\partial_\mu - igA_\mu)\,\phi = D_\mu \phi$$

and the kinetic term for the gauge field $A_\mu$. Therefore

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (\partial_\mu + igA_\mu)\,\phi^\dagger\,(\partial^\mu - igA^\mu)\,\phi \\ & \quad -\mu^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2 \ . \end{aligned}$$

It is difficult to read directly the masses of the particles from this Lagrangian because we have a mixing term $A_\mu \partial^\mu \theta$ between the Goldstone boson $\theta$ and the gauge boson $A_\mu$. It is possible, by a gauge transformation to eliminate the mixing term because we saw how to completely eliminate the Goldstone boson from the Lagrangian earlier. In polar coordinates

$$\phi = \frac{1}{\sqrt{2}}\,\rho\,e^{i\theta}\,, \qquad \phi^\dagger = \frac{1}{\sqrt{2}}\,\rho\,e^{-i\theta}\,.$$

The gauge transformation that eliminates $\theta$ is

$$\phi \to \phi\, e^{-i\theta}$$

for the scalar field, and

$$A_\mu \to A_\mu - \frac{1}{g}\partial_\mu \theta$$

for the gauge field. The Lagrangian becomes

$$\begin{aligned}\mathcal{L} \;&=\; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu + igA_\mu)\rho(\partial^\mu - igA^\mu)\rho \\ &\quad -\frac{\mu^2}{2}\rho^2 - \frac{\lambda}{4}\rho^4 \;.\end{aligned}$$

We have to perform the translation in order to be around the minimum $\rho = \chi + v$, and one can see that the covariant derivative term generates a mass term for the gauge field

$$\frac{1}{2} g^2 v^2 A_\mu A^\mu \ ,$$

thus the gauge field has mass

$$m_A^2 = g^2 v^2$$

and the Goldstone boson has disappeared from the theory.

The choice of gauge where the Goldstone boson vanishes is the *gauge* unit. Note that the number of degrees of freedom of the theory has not changed: Initially we had two real scalar fields and two components of a massless gauge boson. After the gauge transformation we had a single real scalar field and three components of a massive boson.

In general, if the global symmetry group of the Lagrangian is $G$, $H \subset G$ is the invariance group of the vacuum, and $G_W \subset G$ the local gauge symmetry (with $K = H \cap G_W \neq 0$) the broken generators of $G$ can be separated into two categories: $T_K \in K$ are the generators associated with the massive gauge bosons, and the other broken generators correspond to massless Goldstone bosons. The unbroken generator $G_W$ corresponds to massless gauge bosons.

# Lagrangian of the Standard Model

The Lagrangian of the Standard Model is given by

$$\mathcal{L}_{\mathrm{SM}} = \mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{D}} + \mathcal{L}_{\mathrm{Yukawa}} + \mathcal{L}_{\mathrm{Higgs}}$$

and we will examine each part of the Lagrangian and its symmetries.

# The Gauge Sector

The first part of the SM Lagrangian is the kinetic part of the gauge fields:

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4g_1^2} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4g_2^2} W^a_{\mu\nu} W^{a\,\mu\nu} - \frac{1}{4g_3^2} G^A_{\mu\nu} G^{A\,\mu\nu}$$

where $g_1$, $g_2$, $g_3$ are the couplings respectively of the hypercharge, of isotopic spin (isospin), and colour. The tensors in the above equation are

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

for hypercharge, with $B_\mu$ being the boson vector field of the hypercharge $U(1)$.

For the isospin:

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - \epsilon^{abc} W^b_\mu W^c_\nu$$

with $W^a_\mu$ $(a = 1, 2, 3)$ being the vector bosons of the $SU(2)$ weak isospin and $\epsilon^{abc}$ the antisymmetric structure constant of $SU(2)$. For the $SU(3)$ colour group

$$G^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu - f^{ABC} A^B_\mu A^C_\nu$$

where the $A^A_\mu$ $(A = 1, \ldots 8)$ are the gluon fields, and $f^{ABC}$ the antisymmetric structure constants of $SU(3)$.

Note that hypercharge and isospin for the $SU(2) \otimes U(1)$ gauge theory will be broken by the Higgs mechanism to give the unbroken $U(1)$ EM theory (QED), and the broken generators give the three massive vector bosons of the weak interactions.

But more on that later.

# The Dirac Sector

The term $\mathcal{L}_{\mathrm{D}}$ is the Lagrangian for the Dirac fermions, describing the freely moving fermions and the fermionic interactions with gauge bosons.

Recall that weak interactions violate parity, as such, we'll describe the Dirac fermions in terms of Weyl spinors with two-components

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

to highlight that fact.

Quarks and leptons of the SM are written in terms of multiplets $(SU(3)_c,\ SU(2)_w,\ U(1)_y)$ and using only two component spinors of the left-handed type:

$$\begin{aligned} L_i &= \begin{pmatrix} \nu_i \\ e_i \end{pmatrix}_L \sim (1,\,2,\,y_1) \\ \bar{e}_{i\,L} &\sim (1,\,1,\,y_2) \\ Q_i &= \begin{pmatrix} u_i \\ d_i \end{pmatrix}_L \sim (3,\,2,\,y_3) \\ \bar{u}_{i\,L} &\sim (\bar{3},\,1,\,y_4) \\ \bar{d}_{i\,L} &\sim (\bar{3},\,1,\,y_5) \end{aligned}$$

where $i$ is the index which indicates the family.

For now the values $y_1 \ldots y_5$ of the hypercharge shall remain undetermined.

The coupling of fermions to gauge fields is done with covariant derivatives. For gauge fields we will use a notation in terms of matrices

$$\tilde{W}_\mu = \frac{1}{2} W_\mu^a \tau^a \qquad\qquad \tilde{A}_\mu = \frac{1}{2} A_\mu^A \lambda^A$$

with $\tau^a$ being $SU(2)_w$ (Pauli) matrices and $\lambda^A$ those of $SU(3)_c$ (Gell-Mann matrices). In the following we will indicate the Pauli matrices with $\tau^i$ when done in reference to $SU(2)_w$ matrices and with $\sigma^i$ for spin.

The covariant derivatives are defined by

$$\begin{aligned}
D_\mu L_i &= \left(\partial_\mu + i\tilde{W}_\mu + i\frac{y_1}{2}B_\mu\right) L_i \\
D_\mu \bar{e}_i &= \left(\partial_\mu + \frac{i}{2}y_2 B_\mu\right) \bar{e}_i \\
D_\mu Q_i &= \left(\partial_\mu + i\tilde{A}_\mu + i\tilde{W}_\mu + \frac{i}{2}y_3 B_\mu\right) Q_i \\
D_\mu \bar{u}_i &= \left(\partial_\mu - i\tilde{A}^*_\mu + \frac{i}{2}y_4 B_\mu\right) \bar{u}_i \\
D_\mu \bar{d}_i &= \left(\partial_\mu - i\tilde{A}^*_\mu + \frac{i}{2}y_5 B_\mu\right) \bar{d}_i \ .
\end{aligned}$$

The Dirac part of the SM Lagrangian is

$$\mathcal{L}_{\rm D} = \sum_{i=1}^{3} \Big( L_i^\dagger \sigma^\mu D_\mu L_i + \bar{e}_i^\dagger \sigma^\mu D_\mu \bar{e}_i + Q_i^\dagger \sigma^\mu D_\mu Q_i$$
$$+ \bar{u}_i^\dagger \sigma^\mu D_\mu \bar{u}_i + \bar{d}_i^\dagger \sigma^\mu D_\mu \bar{d}_i \Big) \ .$$

Note that the Lagrangian $\mathcal{L}_{\rm Y\mathit{M}} + \mathcal{L}_{\rm D}$ has a symmetry larger than the full Lagrangian of the SM. For the multiplets used above, and given that we have 5 types of fermions, the overall symmetry appears to be $[U(3)]^5$. This symmetry will not be respected in the other parts of the Lagrangian. In particular, in the Yukawa sector.

# Yukawa interactions

The need to introduce Yukawa terms (terms of dimension 4 with two spinors and a scalar field) is due to the impossibility of writing mass terms which are invariant and renormalisable, such as:

$$L^T \sigma_2 \bar{e}_L \ , \qquad Q^T \sigma_2 \bar{u}_L \ , \qquad Q^T \sigma_2 \bar{d}_L \ ,$$

which are not invariant with respect to the weak isospin.

One possible way to construct mass terms which are invariant is to introduce a scalar field that is an isospin doublet, like the Higgs field:

$$H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \sim (1,\, 2,\, y_h) \ ,$$

and construct interaction terms (scalar-fermion-fermion), the Yukawa terms:

$$\begin{aligned}\mathcal{L}_{\text{Yukawa}} &= iY^e_{ij} L^T_i \sigma_2 \bar{e}_{jL} H^* + iY^u_{ij} Q^T_i \sigma_2 \bar{u}_{jL} \tau_2 H \\ &\qquad + iY^d_{ij} Q^T_i \sigma_2 \bar{d}_{jL} H^* + \text{h.}c.\end{aligned}$$

where the $Y_{ij}$ are complex matrices of Yukawa couplings.

Recall that $\tau_i$ denotes the Pauli matrices for the $SU(2)_w$ group and $\sigma_i$ the same Pauli matrices for spin.

Note that after spontaneous symmetry breaking these terms give rise to fermionic mass terms.

The hypercharge conservation imposes the following relations:

$$y_h = y_1 + y_2 = -(y_3 + y_4) = y_3 + y_5$$

and if we fix $y_h = 1$, a choice consistent with anomaly cancellation is:

$$y_1 = -1\ ,\ y_2 = +2\ ,\ y_3 = +1/3\ ,\ y_4 = -4/3\ ,\ y_5 = 2/3\ .$$

The Yukawa couplings $Y_{ij}$ are not all independent because redefinitions of the fields are possible using the global symmetries of $\mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{D}}$.

Note that any complex matrix can be written as:

$$Y^e = U^{e\,T} M^e V^e$$

with $U^e U^{e\dagger} = V^e V^{e\dagger} = 1$ ($U^e$ and $V^e$ being unitary matrices) and $M^e$ a real diagonal matrix. The unitary matrices can be absorbed by a redefinition of fields:

$$L' = U^e L \qquad\qquad \bar{e}'_L = V^e \bar{e}_L$$

without changing the $\mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{D}}$.

For leptons this redefinition makes $\mathcal{L}_{\mathrm{Yukawa}}$ diagonal:

$$i y^e_{ii} L_i^T \sigma_2 \bar{e}_i H^* + \mathrm{h.}c.$$

with

$$M^e = \begin{pmatrix} y^e_{11} & 0 & 0 \\ 0 & y^e_{22} & 0 \\ 0 & 0 & y^e_{33} \end{pmatrix} \; .$$

These Yukawa terms break the global $U(3) \times U(3)$ symmetry and only retain the $U(1)$ invariance with phase

$$L_i \to e^{i\alpha_i} L_i \; , \qquad \bar{e}_i \to e^{-i\alpha_i} \bar{e}_i \; ,$$

the $\alpha_i$ being interpreted as the three leptonic numbers.

The redefinition of quark fields of type *up* and *down* can not be done independently, because the two types of Yukawa interactions for quarks always contain $Q_i$.

Note that if we wanted to have massive neutrinos, we would have the same problem with $L_i$, which would lead to the $V_{PMNS}$ in the same way that this section will lead to $V_{CKM}$.

Writing:

$$Y^u = U^{u\,T} M^u V^u \;, \qquad Y^d = U^{d\,T} M^d V^d \;,$$

and for quarks $\bar{u} \to V^u \bar{u}$ , $\bar{d} \to V^d \bar{d}$

The doublet $Q_i$ can be redefined to eliminate the matrix $U$ and the remaining two Yukawa interaction terms leave matrices in the other couplings:

$$iy^d_{ii} Q^T_i \sigma_2 \bar{d}_i H^* + iy^u_{jj} Q^T_i \sigma_2 \mathcal{V}_{ji} \bar{u}_j \tau_2 H$$

with $\mathcal{V} = U^u U^{d\dagger}$. The matrix $\mathcal{V}$ is unitary and therefore there are now 9 independent parameters instead of 18, by the relationship $\mathcal{V}^\dagger \mathcal{V} = 1$.

Note that $\mathcal{V}_{ji}$ will give rise to $V_{CKM}$. Further simplifications are possible such that we would be left with a matrix containing three parameters (angles) of the matrix, and a phase, all of which are physical!

The only remaining symmetry in the quark sector of the global symmetry is a $U(1)$ phase common to all quarks

$$Q_i \to e^{i\delta} Q_i \qquad \bar{u}_i \to e^{-i\delta} \bar{u}_i \qquad \bar{d}_i \to e^{-i\delta} \bar{d}_i$$

which corresponds to a conserved quantum number, baryon number.

# The Higgs sector

Before we proceed to EWSB, we need to study the final piece of $\mathcal{L}_{SM}$. This will also be the sector responsible for the symmetry breaking $SU(2)_w \otimes U(1)_y \to U(1)_{\rm em}$.

In the previous section we introduced a complex scalar doublet of $SU(2)_w$

$$H = \left( \begin{array}{c} H_1 \\ H_2 \end{array} \right) \sim (1,\, 2,\, 1)\ ,$$

where we now write its Lagrangian as

$$\mathcal{L}_{\rm Higgs} = (D_\mu H)^\dagger\ (D^\mu H) - V(H)$$

with

$$
\begin{aligned}
D_\mu H &= \left(\partial_\mu + i\tilde{W}_\mu + \frac{i}{2} y_h B_\mu\right) H \text{ with } y_h = 1 \\
V(H) &= -\mu^2 H^\dagger H + \lambda (H^\dagger H)^2 \,.
\end{aligned}
$$

The potential $V$ is the broadest possible renormalisable and invariant potential under the $SU(2)_w \otimes U(1)_y$ symmetry.

The invariance of the vacuum is that of the $U(1)_{\mathrm{em}}$, so one component of this pair must be a neutral scalar field of the electric charge.

One can check that our choice of the previous section, $y_h = 1$, is in agreement with this observation. The relationship between the electric charge, hypercharge and isospin is

$$Q_{\rm em} = I_{3w} + \frac{1}{2} y \; .$$

For both components of the doublet Higgs has

$$\begin{aligned} Q_{\rm em}(\phi^+) &= \frac{1}{2} + \frac{1}{2} y_h = 1 \\ Q_{\rm em}(\phi^0) &= -\frac{1}{2} + \frac{1}{2} y_h = 0 \end{aligned}$$

Thus we find the component $\phi^0$ with zero electric charge.

# Spontaneous electroweak symmetry breaking

With the choice of parameters $\mu^2 < 0$ and $\lambda > 0$ the Higgs potential has its minimum on the surface

$$|H|^2_{\mathrm{min}} = -\frac{\mu^2}{2\lambda} = \frac{v^2}{2}$$

with $v^2 = -\mu^2/\lambda$. We will choose the vacuum

$$\langle 0|H|0\rangle = \left( \begin{array}{c} 0 \\ \frac{v}{\sqrt{2}} \end{array} \right)$$

and set the fields around this vacuum as

$$H = \exp\left(\frac{i}{v}\xi_i(x)\sigma_i\right) \left( \begin{array}{c} 0 \\ \frac{v+h(x)}{\sqrt{2}} \end{array} \right) \equiv U(x)H_0 \ ,$$

where we have introduced the fields $\xi_i(x)$ ($i = 1,\ 2,\ 3$) and $h(x)$.

The unitary phase matrix $U(x)$ is a gauge transformation of $SU(2)$ and is a direct result of the unitary gauge. The corresponding gauge transformation on the $SU(2)$ gauge fields lies in studying the covariant derivative

$$\begin{aligned} D_\mu H &= \left(\partial_\mu + i\tilde{W}_\mu + \frac{i}{2}B_\mu\right) U(x) \begin{pmatrix} 0 \\ \frac{v+h(x)}{\sqrt{2}} \end{pmatrix} \\ &= U(x)\, U(x)^\dagger \left(\partial_\mu + i\tilde{W}_\mu + \frac{i}{2}B_\mu\right) U(x) \begin{pmatrix} 0 \\ \frac{v+h(x)}{\sqrt{2}} \end{pmatrix} \\ &= U(x) \left(\partial_\mu + i\tilde{W}'_\mu + \frac{i}{2}B_\mu\right) \begin{pmatrix} 0 \\ \frac{v+h(x)}{\sqrt{2}} \end{pmatrix} , \end{aligned}$$

where the last equality is obtained by taking

$$\tilde{W}'_\mu = -iU(x)^\dagger \partial_\mu U(x) + U(x)^\dagger \tilde{W}_\mu U(x) \; .$$

In this way the matrix $U(x)$ vanishes completely from the Lagrangian:

$$\begin{aligned}\mathcal{L}_{\text{Higgs}} \quad &= \quad \frac{1}{2}\partial_\mu h\,\partial^\mu h + \frac{1}{8}\left(B_\mu - W_{3\mu}\right)\left(B^\mu - W_3^\mu\right)(v+h)^2 \\ &\quad + \frac{1}{8}\left(W_{1\mu} - iW_{2\mu}\right)\left(W_1^\mu + iW_2^\mu\right)(v+h)^2 \\ &\quad + \lambda v^2 h^2 + \lambda v h^3 + \frac{\lambda}{4}h^4 - \lambda\frac{v^2}{4}\ .\end{aligned}$$

You can read the mass term for the Higgs boson as:

$$\lambda v^2 h^2 = \frac{1}{2}2\lambda v^2 h^2 = \frac{1}{2}m_h^2 h^2$$

where we define $m_h^2 = 2\lambda v^2$.

However, it is difficult to read the mass terms of the gauge bosons due to the mixing of terms. We must therefore define linear combinations appropriate to the fields in order to eliminate mixing terms between gauge bosons.

Before this we will reintroduce the coupling constants which were hidden in the fields earlier

$$B_\mu \to g_1 B_\mu \ , \qquad \tilde{W}_\mu \to g_2 \tilde{W}_\mu \ , \qquad \tilde{A}^A_\mu \to g_3 \tilde{A}^A_\mu \ .$$

This will make the $\mathcal{L}_{YM}$ I gave earlier look like the more usual kinetic terms of gauge fields

To find the diagonal form of the masses we will impose in the appropriate area:

$$m_W^2 W_\mu^+ W^{-\mu} \equiv \frac{g_2^2 v^2}{8} \left(W_{1\mu} - iW_{2\mu}\right) \left(W_1^\mu + iW_2^\mu\right)$$

where

$$W_\mu^\pm = \frac{1}{\sqrt{2}} \left(W_{1\mu} \mp iW_{2\mu}\right)$$

and the mass of the two gauge bosons is $m_W^2 = \dfrac{g_2^2 v^2}{4}$

For the neutral gauge bosons the electrical charge imposes a massless linear combination corresponding to the photon as

$$\frac{1}{2}m_Z^2 Z_\mu Z^\mu + \frac{1}{2}\,0\,A_\mu A^\mu \equiv \frac{v^2}{8}\left(g_1 B_\mu - g_2 W_{3\mu}\right)\left(g_1 B^\mu - g_2 W_3^\mu\right)\,.$$

The above equation can be written in terms of matrices

$$\tfrac{1}{2}(Z_\mu,\, A_\mu)\begin{pmatrix} m_Z^2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Z^\mu \\ A^\mu \end{pmatrix} \equiv$$
$$\tfrac{v^2}{8}(W_{3\mu},\, B_\mu)\begin{pmatrix} g_2^2 & -g_1 g_2 \\ -g_1 g_2 & g_1^2 \end{pmatrix}\begin{pmatrix} W_3^\mu \\ B^\mu \end{pmatrix}$$

and the link between the two descriptions is an orthogonal transformation

$$\begin{pmatrix} Z^\mu \\ A^\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_w & -\sin\theta_w \\ \sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} W_3^\mu \\ B^\mu \end{pmatrix}$$

with

$$\cos\theta_w \equiv \frac{g_2}{\sqrt{g_1^2 + g_2^2}} \qquad\qquad \sin\theta_w \equiv \frac{g_1}{\sqrt{g_1^2 + g_2^2}} \ ,$$

where $\theta_w$ is the Weinberg angle.

The mass of the photon is zero and that of the boson $Z^0$ equals

$$m_Z^2 = \frac{v^2}{4}(g_1^2 + g_2^2) \ .$$

A comparison between equations for $Z$ and $W^\pm$ masses gives the relationship, valid at the tree-level

$$\frac{m_W^2}{m_Z^2} = \cos^2\theta_w \ .$$

Recall that the couplings of fermions to gauge fields were given by covariant derivatives in $\mathcal{L}_D$. So if we expand $\mathcal{L}_D$ in terms of our "new" physical gauge fields we get

$$\mathcal{L}_{\rm em} = -ieA_\mu \left(e_L^\dagger \sigma^\mu e_L + e_R^\dagger \bar{\sigma}^\mu e_R\right) ,$$

with the interactions of charged weak currents

$$\mathcal{L}_{\rm cc} = i\frac{g_2}{\sqrt{2}} \left(W_\mu^- \nu_{eL}^\dagger \sigma^\mu e_L + W_\mu^+ e_L^\dagger \sigma^\mu \nu_{eL}\right) ,$$

and neutral weak currents

$$\begin{aligned}\mathcal{L}_{\rm cn} \quad &= \quad i\frac{g_2}{\cos\theta_w} Z_\mu \left[\frac{1}{2}\nu_{eL}^\dagger \sigma^\mu \nu_{eL} - \frac{1}{2} e_L^\dagger \sigma^\mu e_L \right. \\ &\qquad \left. + \sin^2\theta_w \left(e_L^\dagger \sigma^\mu e_L + e_R^\dagger \bar{\sigma}^\mu e_R\right)\right] .\end{aligned}$$

From the previous equations we can see the interactions of charged and neutral currents have the same interaction force.

# Conclusion

The phenomenology of this Higgs boson, and how it couples with the fields is beyond the scope of this lecture series, but I hope shall be discussed elsewhere this week, along with the ongoing efforts to measure these interactions.

For further details on these notes, please feel free to contact me at

alan.cornell@wits.ac.za

Thank you