Physics Honours: Standard Model

Tutorial Sheet 7

Question 1

The properties of the mass matrix can be translated into relations between mass values and mixing angles. Here is an illustrative example of such a model. Consider a simple 2×2 Hermitian fermion mass matrix of the form

$$M = \begin{pmatrix} 0 & a \\ a^* & b \end{pmatrix} .$$

Show that the mixing angle θ which characterises the 2×2 unitary matrix which diagonalises M is related to the mass eigenvalues by

$$\tan\theta = \sqrt{\frac{m_1}{m_2}} .$$

Question 2

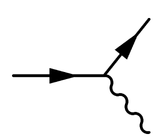
Defining the scalar fields on the vacuum as follows:

$$H = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} + \begin{pmatrix} \phi^+ \\ \frac{h+i\eta}{\sqrt{2}} \end{pmatrix} \equiv H_0 + H' ,$$

find the spectrum of masses of Goldstone bosons and the physical Higgs field in the theory.

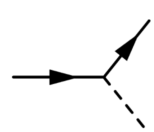
Question 3

Calculate the width of decay $t \to W\, b$ given by the Feynman diagram



and assuming that it takes no account of colour and that we can not determine the spins of the particles (averaged over the initial states and summed over the final states) and neglecting the b quark mass in comparison to the mass of the W boson and the t quark.

Next, redo the calculation by replacing the boson $W^\pm$ by its longitudinal Goldstone boson $\phi^\pm$



and verify that the result is similar to the dominant order of a series expansion in m_W/m_t as provided by the equivalence theorem.

Question 4

The Lagrangian for a complex scalar doublet ϕ is given by

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) + \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2$$

with

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} .$$

Show that if we write the complex fields by the components

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} ,$$

the Lagrangian is explicitly invariant under $O(4)$ transformations.

Then write the Lagrangian according to

$$\pi = (\phi_1,\ \phi_2,\ \phi_4) , \qquad \sigma = \phi_3$$

and with the spontaneous symmetry breaking

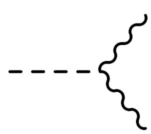
$$\sigma = v + h ,$$

where $v^2 = \mu^2/\lambda$ is the value of the vacuum field σ. Using the Lagrangian, based on fields h and π, calculate the decay widths $h \to \pi^+ \pi^-$ and $h \to \pi^0 \pi^0$

where

$$\pi^\pm = \frac{1}{\sqrt{2}} (\pi_1 \mp i\pi_2) , \qquad \pi^0 = \pi_3 .$$

Show that the calculation of the decay width of the standard model $h \to W^+ W^-$



with $M_h \gg M_W$, gives the same result as the calculation for $h \to \pi^+ \pi^-$, in agreement with the equivalence theorem.

Answers

Question 1

The mass matrix can be diagonalised by an orthogonal transformation

$$SMS^\dagger = M_d = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \quad \text{or} \quad M = S^\dagger M_d S$$

with

$$S = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} .$$

From $M_{11} = 0$, we get

$$S^*_{1i}(M_d)_{ij}S_{j1} = 0 \quad \text{or} \quad (\cos^2\theta)m_1 - (\sin^2\theta)m_2 = 0$$

or

$$\tan\theta = \sqrt{\frac{m_1}{m_2}} .$$

Remark: Attempts to relate the Cabbibo angle to the strange and down quark masses have been carried out along such approaches.

Question 2

We obtain

$$|H|^2 = \frac{v^2}{2} + vh + |\phi^+|^2 \frac{h^2+\eta^2}{2}$$

and the Higgs potential becomes

$$V = -\frac{\mu^4}{4\lambda} + \lambda v^2 h^2 + 2\lambda v h \left(|\phi^+|^2 + \frac{h^2+\eta^2}{2}\right) + \lambda \left(|\phi^+|^2 + \frac{h^2+\eta^2}{2}\right)^2$$

allowing us to read the masses

$$\begin{aligned} m_\eta^2 &= m_{\phi^+} = m_{\phi^-} = 0 \\ m_h^2 &= 2\lambda v^2 = -2\mu^2 \end{aligned}$$

with $\phi^- = \phi^{+\dagger}$. We therefore have a physical scalar h with mass (the Higgs field) and three massless Goldstone bosons, which can be eliminated by a gauge transformation.

Question 3

The matrix element for the top quark decay is

$$M = \frac{ig}{2\sqrt{2}}\, \bar{u}(q)\, \gamma^\mu\, (1-\gamma_5)\, u(p)\, \epsilon^*_\mu(k)\, V_{tb}$$

where p is the momentum of the top quark, q that of the b quark and k that of the W boson with polarisation $\epsilon^*(k)$. Subsequently we will not indicate the mixture V_{tb} since its numerical value is $\simeq 1$. The modulus squared of the matrix element M with a sum over final states and an average over initial states gives

$$|\bar{M}|^2 = \frac{g^2}{2}\, (q^\mu p^\nu + q^\nu p^\mu - g^{\mu\nu} p\cdot q) \sum_{\text{pol}} \epsilon^*_\mu(k)\, \epsilon_\nu(k)$$

where

$$\sum_{\text{pol}} \epsilon^*_\mu(k)\, \epsilon_\nu(k) = -g_{\mu\nu} + \frac{k_\mu k_\nu}{m_W^2} .$$

The contraction of indices gives

$$|\bar{M}|^2 = \frac{g^2}{2}\left(p\cdot q + \frac{2\,k\cdot q\;k\cdot p}{m_W^2}\right) .$$

In the limit $m_b = 0$ scalar products are given by

$$\begin{aligned} 2\,q\cdot p &= 2\,q\cdot k = m_t^2 - m_W^2 \\ 2\,k\cdot p &= m_t^2 + m_W^2 \end{aligned}$$

and

$$|\bar{M}|^2 = \frac{g^2}{4}\frac{m_t^4}{m_W^2}\left(1-\frac{m_W^2}{m_t^2}\right)\left(1+2\frac{m_W^2}{m_t^2}\right) .$$

The decay width can be calculated using the formulas in the appendix

$$\begin{aligned} \frac{d\Gamma}{d\Omega} &= \frac{1}{32\pi^2}\,|\bar{M}|^2\,\frac{|p_1|}{m_t^2} \\ |p_1| &= \frac{1}{2m_t}\sqrt{[m_t^2-(m_W+m_b)^2]\,[m_t^2-(m_W-m_b)^2]} \\ &\simeq \frac{m_t}{2}\left(1-\frac{m_W^2}{m_t^2}\right) . \end{aligned}$$

The total width is obtained by integrating over the differential width

$$\Rightarrow \Gamma = \frac{g^2}{64\pi}\frac{m_t^3}{m_W^2}\left(1-\frac{m_W^2}{m_t^2}\right)^2\left(1+2\frac{m_W^2}{m_t^2}\right) .$$

The calculation with the longitudinal part, $\phi^\pm$, gives

$$M = -i\frac{g\,m_t}{2\sqrt{2}\,m_W}\;\bar{u}(q)\,(1+\gamma_5)\,u(p)$$

whose modulus squared is

$$|\bar{M}|^2 = \frac{g^2\,m_t^2}{2\,m_W^2}\;q\cdot p .$$

The total width is

$$\Gamma = \frac{g^2}{64\pi}\frac{m_t^3}{m_W^2}$$

which is comparable with the exact result. One can see that the result of the dominant order of the series expansion of m_W/m_t is the same.

Question 4

Our expression for ϕ in terms of $\phi_1\,,\ldots,\phi_4$ allows us to write

$$\begin{aligned} \phi^\dagger\phi &= \frac{1}{2}\left(\phi_1^2+\phi_2^2+\phi_3^2+\phi_4^2\right) = \frac{1}{2}\left(\phi\cdot\phi\right) \\ \partial_\mu\phi^\dagger\partial^\mu\phi &= \frac{1}{2}\left(\partial_\mu\phi\cdot\partial^\mu\phi\right) \end{aligned}$$

where $\phi = (\phi_1,\,\phi_2,\,\phi_3,\,\phi_4)$ is a vector of $O(4)$. The Lagrangian can be written according to the scalar product of this vector and its derivatives

$$\mathcal{L} = \partial_\mu\phi\cdot\partial^\mu\phi + \frac{\mu^2}{2}\left(\phi\cdot\phi\right) - \frac{\lambda}{4}\left(\phi\cdot\phi\right)^2 .$$

which is invariant under transformations of $O(4)$.

In the new variables π and σ we have

$$\phi\cdot\phi = \pi^2+\sigma^2$$

and the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu \pi)^2 + (\partial_\mu \sigma)^2\right] + \frac{\mu^2}{2}\left(\pi^2+\sigma^2\right) - \frac{\lambda}{4}\left(\pi^2+\sigma^2\right)^2 .$$

In the broken symmetry $\sigma = v + h$ the scalar potential has the form

$$\begin{aligned} V &= -\frac{\mu^2}{2}\left(\pi^2+\sigma^2\right) + \frac{\lambda}{4}\left(\pi^2+\sigma^2\right)^2 \\ &= \frac{1}{2}\, 2\lambda v^2\, h^2 + \lambda v\, h\, \left(\pi^2+h^2\right) + \frac{\lambda}{4}\left(\pi^2+h^2\right)^2 \end{aligned}$$

and we can read directly from the Lagrangian the mass of the Higgs boson

$$m_h^2 = 2\lambda v^2 .$$

The three fields π remain massless. The Lagrangian, using the mass of the Higgs from the previous formula, is

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu \pi)^2 + (\partial_\mu h)^2\right] - \frac{1}{2} m_h^2 h^2 - \frac{m_h^2}{2v}\, h\, \left(\pi^2+h^2\right) - \frac{m_h^2}{8v^2}\left(\pi^2+h^2\right)^2 .$$

The couplings of the Higgs boson to π^+, π^- and π^0 are obtained using our earlier equation, which gives $\pi^2 = 2\pi^+\pi^- + \pi^0\pi^0$ and for the coupling to the Higgs boson, $h\pi^+\pi^-$ and $h\pi^0\pi^0$, yields im_h^2/v. The calculation of the corresponding widths gives

$$\begin{aligned} \Gamma(h \to \pi^+\,\pi^-) &= \frac{m_h^3 G_F}{8\sqrt{2}\,\pi} , \\ \Gamma(h \to \pi^0\,\pi^0) &= \frac{m_h^3 G_F}{16\sqrt{2}\,\pi} . \end{aligned}$$

The amplitude hW^+W^- in the Standard Model is

$$M = -igm_W\,(\epsilon_1 \cdot \epsilon_2)$$

with the modulus square of the amplitude, with the sum over the polarisations of W,

$$\begin{aligned} \sum |M|^2 &= g^2 m_W^2 \sum (\epsilon_1 \cdot \epsilon_2)^2 = g^2 m_W^2 \left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{m_W^2}\right)\left(-g^{\mu\nu} + \frac{k'^\mu k'^\nu}{m_W^2}\right) \\ &= g^2 m_W^2 \left(2 + \frac{(k\cdot k')^2}{m_W^4}\right) \end{aligned}$$

where k and k' are the momenta of the two W. Using $m_h^2 = (k+k')^2$ one can calculate the scalar product

$$k \cdot k' = \frac{1}{2}\left(m_h^2 - 2m_W^2\right)$$

and the modulus square of the amplitude is

$$\sum |M|^2 = \frac{2G_F}{\sqrt{2}} m_h^2 \left(1 - 4\frac{m_W^2}{m_h^2} + 12\frac{m_W^4}{m_h^4}\right) .$$

The width can be calculated using the formulas in the appendix:

$$\Gamma(h \to W^+W^-) = \frac{m_h^3 G_F}{8\sqrt{2}\,\pi}\left(1 - 4\frac{m_W^2}{m_h^2} + 12\frac{m_W^4}{m_h^4}\right)\left(1 - 4\frac{m_W^2}{m_h^2}\right)^{1/2} .$$

These results, with $m_h \gg m_W$, is in agreement with the results in the scalar theory, as provided by the equivalence theorem.