Quantum measurements along accelerated world-lines

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Abstract. In this research, we are working with a formalism for quantum measurements that takes special relativity into account. The ultimate goal is to modify this framework to work with more general space-times rather than just Minkowski space-time and determine how the metric would affect quantum entanglement by doing a calculation of Bell's Theorem in curved space-time. As a first step in that direction, in this paper, we calculate the case for quantum measurements along an accelerated world line by solving the Schwinger-Tomonaga equation.

1. Introduction

Most of modern physics can be described either within the framework of general relativity or within the framework of quantum mechanics. General relativity describes one of the four fundamental forces, gravitation, as a warpage of space-time whereas the other three, namely the electromagnetic, the strong and weak nuclear forces can be adequately described within the framework of quantum mechanics. However, combining general relativity with quantum mechanics in order to formulate a theory of quantum gravity has proven difficult. The ultimate goal of the current work is to modify a framework of relativistic quantum mechanics, though not full quantum field theory, such that it includes metrics other than the Minkowski metric as the space-time background. This relativistic framework was formulated by Breuer and Petruccione [1] [2]. The goal is also to determine what the effect of the space-time background, if any, is on the measurements of entangled particles.

This paper summarises the framework, both non-relativistic and relativistic. In the non-relativistic case, the framework is a statistical multi-particle formalism of quantum mechanics which is formulated in terms of probabilities and allows for interactions between different quantum particles. In the relativistic case, we work with the Schwinger-Tomonaga equation instead of the Schröedinger equation which, in this case, is a functional differential equation and the state-vectors are taken as functionals that take as input, space-like hypersurfaces in space-time. The ultimate goal of this research is to extend this special relativistic framework to work with curved space-time backgrounds and then find out what, if any, effect the metric itself has on the phenomenon of quantum entanglement. That means to work with more generalised space-time backgrounds rather than just Minkowski space-time.

2. Quantum Measurements

In an ideal measurement in quantum mechanics, if a property B with corresponding projection operator $E(B)$ is measured on a quantum statistical ensemble $\mathcal{E}$ described by density matrix ρ, then after the measurement, we find that the density matrix ρ' which describes the ensemble $\mathcal{E}'$ that consists of the systems for which the property B is found to be true is given by

$$\rho' = \frac{E(B)\rho E(B)}{\mathrm{tr}\{E(B)\rho E(B)\}}, \tag{1}$$

where the projection operator $E(B)$ is defined in terms of the spectral family of $\hat{R} = \int_{-\infty}^{\infty} r dE_r$ by $P_R(B) = \langle \psi | E(B) | \psi \rangle$ (see [1], pp 59-62). Equation (1) is called the von Neumann-Lüders projection postulate [3] [4]. The above describes the ideal measurement of the projection $E(B)$ derived from the spectral family of $\hat{R}$ but in practice one can only measure an approximation that involves the finite resolution of the detector. If we consider a measurement scheme which yields a set $\mathcal{M}$ of possible outcomes $m \in \mathcal{M}$, then the von Neumann-Lüders projection postulate (1) can be generalised as follows:

(i) The measurement outcome m represents a classical random number with probability distribution

$$P(m) = \mathrm{tr}\{F_m \rho\}, \tag{2}$$

where F_m is a positive operator, called the effect, which satisfies the normalisation condition

$$\sum_{m \in \mathcal{M}} F_m = I, \tag{3}$$

such that the probability $P(m)$ is also normalised as

$$\sum_{m \in \mathcal{M}} P(m) = 1. \tag{4}$$

(ii) In the case of a selective measurement, the sub-ensemble of the systems for which the outcome m has been found to be described by the density matrix

$$\rho'_m = P(m)^{-1} \Phi_m(\rho), \tag{5}$$

where $\Phi_m = \Phi_m(\rho)$ is a positive super-operator, called an operation, and it maps positive operators to positive operators. We also assume that the operation Φ_m obeys the condition

$$\mathrm{tr}\Phi_m(\rho) = \mathrm{tr}\{F_m \rho\}. \tag{6}$$

Equation (6) together with equation (2) yields the normalisation

$$\mathrm{tr}\rho'_m = P(m)^{-1} \mathrm{tr}\Phi_m(\rho) = 1. \tag{7}$$

(iii) The density matrix for the corresponding non-selective measurement is given by

$$\rho' = \sum_{m \in \mathcal{M}} P(m)\rho'_m = \sum_{m \in \mathcal{M}} \Phi_m(\rho), \tag{8}$$

which is normalised according to equations (6) and (3) as

$$\mathrm{tr}\rho' = \sum_{m \in \mathcal{M}} \mathrm{tr}\Phi_m(\rho) = \sum_{m \in \mathcal{M}} \mathrm{tr}\{F_m \rho\} = \mathrm{tr}\rho = 1. \tag{9}$$

An important measurement scheme which can be treated within the framework of the generalised theory of quantum measurements above is the concept of an indirect measurement. Instead of directly measuring the system, in an indirect measurement, we perform the measurement on what we will call a quantum probe that has interacted with the system at some point. The aim of the indirect measurement scheme we're going to describe here is to obtain information on the state of the object that we want to measure by performing measurements on the probe ([1], pp 96). This indirect measurement scheme can be considered to be consisting of three elements. The three elements are the quantum system to be measured, called the quantum object and has a Hilbert space $\mathcal{H}_O$, the quantum probe with Hilbert space $\mathcal{H}_P$ and a classical apparatus by which a measurement is performed on the quantum probe following it's interaction with the quantum object. Thus for an ideal measurement, we have three requirements. The first requirement is that prior to the interaction, and at time $t = 0$, the probe is prepared in a well defined quantum state ρ_P while the quantum object is in a state ρ_O. The second requirement is that the the measurement takes place after the interaction is over. So, the interaction between the probe and object may start at time $t = 0$ and end at time $t = \tau > 0$ but the measurement may only take place after the interaction has ended. The third requirement is that the measurement on the probe by the classical apparatus can be described as an ideal measurement by the von Neumann-Lüders projection postulate described above.

At the initial time $t = 0$, the density matrix of the combined system consisting of both probe and object is given by the tensor product $\rho_O \otimes \rho_P$ in the total Hilbert space given by $\mathcal{H} = \mathcal{H}_O \otimes \mathcal{H}_P$. The Hamiltonian of the total system is given by

$$H(t) = H_O + H_P + H_I(t), \tag{10}$$

where H_O and H_P, describe the free evolution of the object and probe respectively. The $H_I(t)$ term describes the evolution due to the interaction between the object and the probe. Outside the interaction time interval, $[0, \tau]$, the term $H_I(t)$ vanishes. The time evolution over this time interval according to the Schröedinger equation

$$i\frac{d}{dt}\left|\psi(t)\right\rangle = H(t)\left|\psi(t)\right\rangle, \tag{11}$$

can be described by a unitary operator called the time-evolution operator. It is given by

$$U \equiv U(\tau, 0) = T_{\leftarrow} \exp\left[-i\int_0^\tau dt H(t)\right], \tag{12}$$

where $T_{\leftarrow}$ is the chronological time-ordering operator that order the products of time dependent operators from right to left, in the direction of the arrow with earlier times on the right and the later times on the left, and we have used units such that $\hbar = 1$. If we assume $\rho(0) = \rho_O \otimes \rho_P$ to be the initial density matrix, then in terms of the time-evolution operator, the density matrix at time τ is given by

$$\rho(\tau) = U(\rho_O \otimes \rho_P)U^\dagger. \tag{13}$$

If we now assume that at time τ, the classical apparatus measures a probe observable $\hat{R} = \sum_m r_m \left|\varphi_m\right\rangle \left\langle\varphi_m\right|$, then we can obtain the indirect measurement by applying the von Neumann-Lüders projection postulate to the measurement of the probe and then by introducing the spectral decomposition of the density matrix of the probe, $\rho_P = \sum_k p_k \left|\phi_k\right\rangle \left\langle\phi_k\right|$, we can derive a measurement operation on the quantum object of the form ([1], pp 97-99)

$$\Phi_m(\rho_O) = \sum_k \Omega_{mk}\rho_O\Omega_{mk}^\dagger, \tag{14}$$

where $\Omega_{mk} = \sqrt{p_k}\left\langle\varphi_m \mid U \mid \phi_k\right\rangle$. When using these operators, the effect is given by $F_m = \sum_k \Omega_{mk}^\dagger \Omega_{mk}$ and so both the operaation and effect has the form of the representation theorem (see equations 2.157 and 2.158 on page 89 of Breuer and Petruccione [1], pp 97-99).

3. Relativistic Quantum Mechanics

In the relativistic framework, everything is re-formulated in terms of four-vectors with a Minkowski background. Here the four-vector $x^\mu = (x^0, \vec{x})$ represents the space-time coordinates of an event x in Minkowski space-time and we use the symbol $\eta_{\mu\nu}$ to represent the Minkowski metric. The Interaction picture representation of the Schröedinger equation ([1], pp 112-115) is used with the Interaction Hamiltonian defined as $H_I = \int \mathcal{H}(t, \vec{x}) d^3x$, where $\mathcal{H}(t, \vec{x})$ is the Hamiltonian density. In a given fixed coordinate system, the vector $|\Psi(t)\rangle$ gives the state of a quantum mechanical system at each time $x^0 = t$ and so allows the evaluation of expectation values for all observables which are localised on the hypersurface $x^0 = t =$ constant in Minkowski space-time. In order to get a Lorentz invariant generalisation of this concept, consider a state vector associated with a three-dimensional space-like hypersurface σ which is defined as a manifold in Minkowski space that extends to infinity in all directions. Consider further that at each point $x \in \sigma$ on the hypersurface, there exists a unit, timelike normal vector $n^\mu(x)$ satisfying the normalisation $n_\mu(x) n^\mu(x) = 1, n^0(x) \geqslant 1$. The state vector then becomes a functional $|\Psi\rangle = |\Psi(\sigma)\rangle$ in the space of all such hypersurfaces. The same is true of the density matrix of the system which is given as the functional $\rho = \rho(\sigma)$. The generalisation of the Schröedinger equation in this relativistic framework is thus a functional differential equation and is given by the Schwinger-Tomonaga equation [5] [6] [7] [8], which is given by

$$\frac{\delta \left|\Psi(\sigma)\right\rangle}{\delta\sigma(x)} = -i\mathcal{H}(x) \left|\Psi(\sigma)\right\rangle . \tag{15}$$

In direct analogy to partial differential equations, the Schwinger-Tomonaga equation is subject to the integrability condition

$$\frac{\delta^2 \rho(\sigma)}{\delta\sigma(x)\delta\sigma(y)} - \frac{\delta^2 \rho(\sigma)}{\delta\sigma(y)\delta\sigma(x)} = \left[\left[\mathcal{H}(x), \mathcal{H}(y)\right], \rho(\sigma)\right] = 0, \tag{16}$$

where the points x and y are located on the same hyper surface σ. This integrability condition is a direct consequence of the requirement of the micro-causality of the Hamiltonian density which states that $\mathcal{H}(x)$ and $\mathcal{H}(y)$ must commute if x and y are space like separated, i.e $[\mathcal{H}(x), \mathcal{H}(y)] = 0$ for $(x-y)^2 < 0$. This integrability condition insures that the Schwinger-Tomonaga equation has a unique solution $\rho(\sigma)$ once one has chosen an appropriate initial density matrix $\rho(\sigma_0)$ for an initial hypersurface σ_0. This solution is normally given as

$$\rho(\sigma) = U(\sigma, \sigma_0)\rho(\sigma_0)U^\dagger(\sigma, \sigma_0), \tag{17}$$

where $U(\sigma, \sigma_0)$ is the generalisation of the unitary time-evolution operator given by

$$U(\sigma, \sigma_0) = T_{\leftarrow} \exp\left[-i\int_{\sigma_0}^{\sigma} d^4x \mathcal{H}(x)\right], \tag{18}$$

where $T_{\leftarrow}$ is the chronological time-ordering operator, as usual where the time-ordering is ordered by the hypersurfaces of the foliation from the initial hypersurface σ_0 on the right up to the hypersurface at time τ given by $\sigma(\tau)$ on the left.

A foliation of Minkowski space is defined as a smooth one-parameter family $\mathcal{F} = \{\sigma(\tau)\}$ of space like hypersurfaces $\sigma(\tau)$ with the property that each space-time point x is located on precisely one hypersurface of the family. A given foliation $\sigma(\tau)$ gives rise to a corresponding family of state vectors $|\Psi(\tau)\rangle = |\Psi(\sigma(\tau))\rangle$. The Schwinger-Tomonaga equation (15) can then be re-formulated as an integral equation

$$\left|\psi(\tau)\right\rangle = \left|\Psi(0)\right\rangle - i\int_{\sigma_0}^{\sigma(\tau)} d^4x \mathcal{H}(x) \left|\Psi(\sigma_x)\right\rangle , \tag{19}$$

where we have denoted $\sigma_x = \sigma(\tau)$ for exactly one parameter value τ.

The hypersurfaces $\sigma(\tau)$ of a foliation can be defined with the help of an implicit equation of the form $f(x,\tau) = 0$, where $f(x,\tau)$ is a smooth scalar function. With an appropriate normalisation of f, the unit normal vector can be assumed to be given by $n_\mu(x) = \frac{\partial f(x,\tau)}{\partial x^\mu}$. Consider two infinitesimally separated hypersurfaces corresponding to two parameter values τ and $\tau + d\tau$. Then $d\left|\Psi(\tau)\right\rangle = \left|\Psi(\tau + d\tau)\right\rangle - \left|\Psi(\tau)\right\rangle$, which according to equation (19) is therefore

$$d\left|\Psi(\tau)\right\rangle = -i\int_{\sigma(\tau)}^{\sigma(\tau+d\tau)} d^4x\mathcal{H}(x)\left|\Psi(\tau)\right\rangle. \tag{20}$$

The four-volume element d^4x can be re-written as $d^4x = d\sigma(x)\left|n_0\frac{\partial x_0}{\partial\tau}\right|d\tau = d\sigma(x)\left|\frac{\partial f}{\partial\tau}\right|d\tau$. Substituting this into equation (20) and dividing both sides by $d\tau$, we get

$$\frac{d}{d\tau}\left|\Psi(\tau)\right\rangle = -i\int_{\sigma(\tau)} d\sigma(x)\left|\frac{\partial f}{\partial\tau}\right|\mathcal{H}(x)\left|\Psi(\tau)\right\rangle \equiv -iH(\tau)\left|\Psi(\tau)\right\rangle. \tag{21}$$

For a particular example, let's consider an observer O moving along a straight world line $y(\tau) = n\tau$ with constant velocity $\vec{v}$ such that $n = \frac{dy}{d\tau} = (\gamma, \gamma\vec{v})$, where $\gamma = \frac{1}{\sqrt{1-|\vec{v}|^2}}$. Here we can see that n is the four-velocity of O. Here, the parameter τ is the proper time of the observer O, or the time measured by a clock carried along the world line $y(\tau)$ by O. At each fixed value of τ, the time axis in the rest frame of observer O is in the direction of the unit vector n while the instantaneous three-space at τ is given by the flat, spacelike hypersurface $\sigma(\tau)$ which is orthogonal to n and contains the point $y(\tau)$. So, the function f is then defined as

$$f(x,\tau) \equiv n(x - y(\tau)) \equiv nx - \tau = 0. \tag{22}$$

We see that the hypersurface $\sigma(\tau)$ consists of all the space-time points x with which the observer O assigns the same time coordinate τ. Since $\left|\frac{\partial f}{\partial\tau}\right| = |-1| = 1$, equation (21) then becomes

$$\frac{d}{d\tau}\left|\Psi(\tau)\right\rangle = -i\int_{\sigma(\tau)} d\sigma(x)\mathcal{H}(x)\left|\Psi(\tau)\right\rangle \equiv -iH(\tau)\left|\Psi(\tau)\right\rangle. \tag{23}$$

In the coordinate system where the normal vector n coincides with the time axis, i.e $n^\mu = (1,0,0,0)$, then equation (23) becomes identical to the Schröedinger equation.

On the other hand, the term $\left|\frac{\partial f}{\partial\tau}\right| \neq 1$ in general, equation (23) only applies if the observer O travels along a straight world-line with constant four-velocity n.

As discussed in the Abstract, before we extend the formalism to the case of general space-times, the first step is to calculate the Schwinger-Tomonaga equation for uniformly accelerated world lines. If we allow for acceleration such that $n(\tau)$ varies in time, then we have to include a different value for $\left|\frac{\partial f}{\partial\tau}\right|$ in general. To take another example, let's consider the case of the observer O accelerating uniformly with respect to an observer O'. In other words, the reference frame of the observer O is such that there is a constant non-zero proper acceleration $\vec{a}$ as felt by observer O. If we again assume that the unit normal vector $n(\tau)$ is the four-velocity, then the four-acceleration is given by $a^\mu(\tau) = \frac{dn^\mu(\tau)}{d\tau}$. Now, since the four-velocity is always normalised as $n^\mu(\tau)n_\mu(\tau) = \eta_{\mu\nu}n^\mu(\tau)n^\nu(\tau) = 1$, we also have $\frac{d}{d\tau}(n_\mu(\tau)n^\mu(\tau)) = 2n_\mu(\tau)a^\mu(\tau) = 0$. Therefore any four-acceleration is orthogonal to the corresponding four-velocity at a given proper time τ, i.e. $a^\mu(\tau)n_\mu(\tau) = 0$ at all times τ. In the observer frame O, the unit normal vector n coincides with the time axis. Therefore the four-velocity of O is $n^\mu = (1,0,0,0)$ when measured with respect to it's own reference frame, while the four-acceleration is given by $a^\mu = (0,\vec{a})$.

Noting that the magnitude of the four-acceleration as measured in the observer frame O is $|(0,\vec{a})| = |\vec{a}| = a$, the only way to make sure that the acceleration of O remains uniform is to ensure that the magnitude of the four-acceleration remains constant at a. Using the fact that a^μ and n^μ are orthogonal, we obtain for the four-acceleration,

$$a^\mu(\tau) = (a\sinh(a\tau), \frac{a}{|\vec{a}|}\cosh(a\tau)\vec{a}), \tag{24}$$

and for the four-velocity,

$$n^\mu(\tau) = (\cosh(a\tau), |\vec{a}|^{-1}\sinh(a\tau)\vec{a}), \tag{25}$$

such that $a^\mu a_\mu = a^2(\sinh^2(a\tau) - \cosh^2(a\tau)) = -a^2$, $a^\mu n_\mu = 0$ and $n^\mu n_\mu = \cosh^2(a\tau) - \sinh^2(a\tau) = 1$. The observer therefore follows a hyperbolic world-line $y(\tau) = (a^{-1}\sinh(a\tau), a^{-1}|\vec{a}|^{-1}\cosh(a\tau)\vec{a})$ and the function f hypersurface is given by $f(x,\tau) \equiv n(\tau)(x - y(\tau)) = n(\tau)x = 0$. Therefore $\left|\frac{\partial f}{\partial \tau}\right| = |a^\mu(\tau)x_\mu|$ and so the Schwinger-Tomonaga equation (21) for the case of an accelerated world line now becomes

$$\frac{d}{d\tau}|\Psi(\tau)\rangle = -i\int_{\sigma(\tau)} d\sigma(x)\, |a^\mu(\tau)x_\mu|\, \mathcal{H}(x)\, |\Psi(\tau)\rangle\,. \tag{26}$$

4. Conclusion

The main part of this paper provides an overview of a framework for special relativistic quantum mechanics, although it is not full quantum field theory. It describes a statistical interpretation of non-relativistic quantum mechanics for multi-particle systems and then proceeds to use that interpretation to describe a theory of quantum measurement. The framework was then generalised to a framework for special relativistic quantum mechanics. This framework was initially formulated by Breuer and Petruccione [1].

The main part of this paper is devoted to the description of non-relativistic measurements. In addition, we have derived the relativistic evolution equation for a closed quantum system with respect to a reference frame moving at constant velocity and a reference frame with uniform acceleration. In future we will describe the projection measurements in a relativistic setting. Based on this and the description of unitary evolution, one could attempt a relativistic description of generalised measurements, all of which could be realised as indirect measurements. We will also try to derive Bell-state measurements within the framework before attempting to introduce more general metrics into the framework other than the Minkowski metric.

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