Defects in the traditional analogy between the dipolar structure of a circular current and a simple electric dipole's

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Abstract. It is shown that when a circular current is resolved into merged distributions of distinct Cartesian x and y component line current elements, each distribution is a complete magnetic dipole that selectively creates *like* Cartesian components of the magnetic torque and *azimuthal* magnetic vector potential, plus only the magnetic field's *other* Cartesian components. All these are expressible in terms of a distribution's *own* magnetic dipolar moment, which is traditionally attributed to the *whole* circular current. In contrast a simple electric dipole aligned on the z-axis, creates its x and y electric torque components, its full cylindrically symmetric electric field and the electric scalar potential, all of which are expressible in terms of the sole electric dipolar moment. Each magnetic or electric Cartesian torque component is expressible as a cross product of a distribution's dipolar moment and one Cartesian field component parallel to an exclusive Cartesian plane perpendicularly bisecting the mutually parallel intra-dipolar displacements, while the distribution's corresponding potential vanishes in that plane. Under such special conditions, tradition compares one *surviving* Cartesian component of the magnetic torque or of the magnetic vector potential to respectively the electric dipole's combined x and y torque components or the whole scalar potential. Seemingly from this and the equality of the magnetic dipolar moments of the two component distributions of the cylindrically symmetric circular current, tradition *incorrectly* defines either of these magnetic dipolar moments as that of the *entire* circular current.

1. Introduction

As a follow up on the earlier paper [1], we show that the traditional analogy of the magnetic dipolar structure of a circular current to that of a simple electric dipole consisting of separated electric scalar charges of identical size but opposite signs has many short comings. This is done by evaluating the dipolar moments, the torques in external fields, the dipolar magnetic vector and electric scalar potentials and their related magnetic and electric fields. To begin with an electric current element is a vector resolvable into perpendicular components, unlike an elemental electric scalar charge which can never be similarly resolved. With respect to their dipolar alignment vectors, magnetic dipolar moments are normal whereas electric dipolar moments are collinear. A magnetic torque is a triple vector product with Cartesian components due to equally perpendicular electric current components. An electric torque is a duo vector product with Cartesian components due to the same electric scalar charge.

Here an elemental current is depicted as an elemental magnetic vector charge since this is more consistent with its nature when contrasted with the elemental electric scalar charge as sources of respective magnetic vector and electric scalar potentials and related fields.

2. Moments of and torques on Cartesian magnetic and electric dipoles

On a circle of radius ρ lying in the xy-plane and centred at the origin O in figure 1(a), an azimuthal line elemental magnetic *vector* charge at point P_j in the jth quadrant is

$$d\mathbf{Q}_j = \mathbf{I}_j \mu_0 d\ell_j \equiv \hat{\boldsymbol{\varphi}}_j I \mu_0 \rho \, d\phi_j = \left(-\hat{\mathbf{x}}\sin\phi_j + \hat{\mathbf{y}}\cos\phi_j \right) I \mu_0 \rho \, d\phi_j, \quad j=1,2,3,4 \qquad (1)$$

where $\hat{\boldsymbol{\varphi}}_j I \mu_0$ is the line magnetic *vector* charge density and $\phi_j = \phi + \frac{\pi}{2}(j-1)$, as the position vector $\boldsymbol{\rho}_j$ is at an angle $0 \le \phi \le \frac{\pi}{2}$ to the x-axis. Hence the Cartesian magnetic vector charge components are

$$d\mathbf{Q}_{j_x} = \pm\hat{\mathbf{x}}dQ_a = \pm\hat{\mathbf{x}}I\mu_0\rho\sin\phi\,d\phi \qquad (2a)$$

$$d\mathbf{Q}_{j_y} = \pm\hat{\mathbf{y}}dQ_b = \pm\hat{\mathbf{y}}I\mu_0\rho\cos\phi\,d\phi \qquad (2b)$$
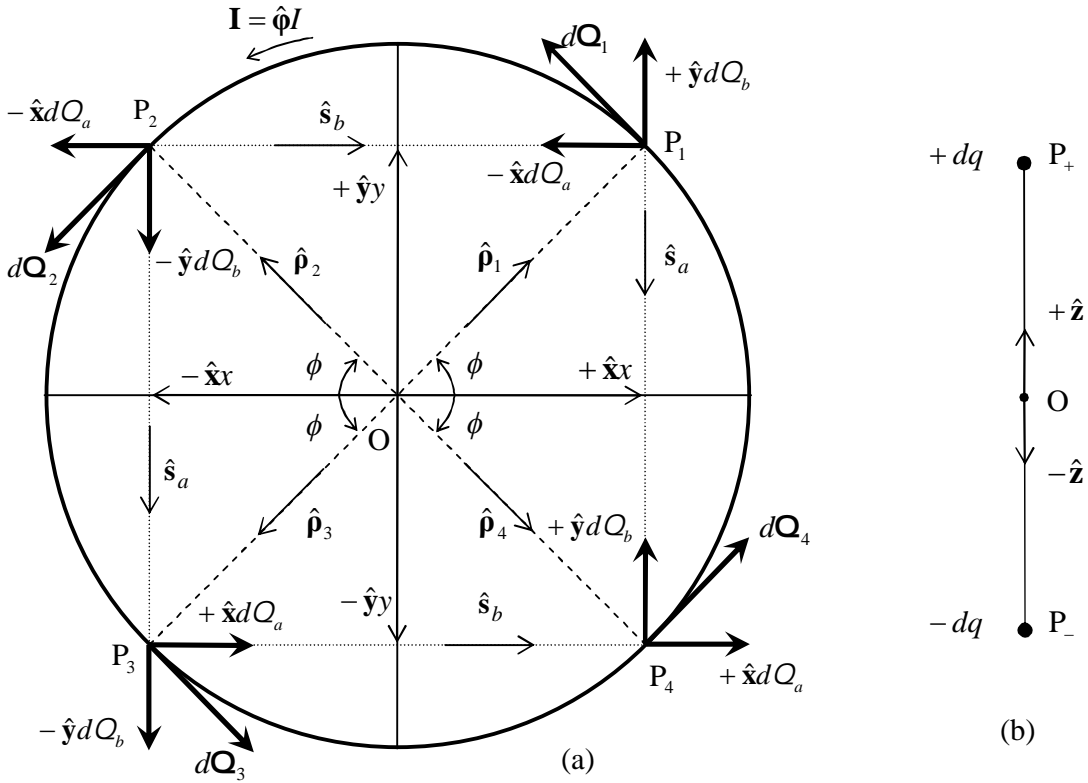

Figure 1. Pairing separated elemental entities (same magnitude, opposite sign) into dipoles: (a) Cartesian magnetic *vector* charges on a circle and (b) electric *scalar* charges on a z-axis.

Pairing the Cartesian components at P_2 with the matching but oppositely directed components at P_1 and P_3 constitutes the a and b Cartesian elemental magnetic dipoles.

While in figure 1(b) at the points P_+ and P_-, of axial position vectors $\mathbf{z}_+ = +\hat{\mathbf{z}}z$ and $\mathbf{z}_- = -\hat{\mathbf{z}}z$ on the z-axis, the line elemental electric *scalar* charges (forming an elemental electric dipole) are

$$+dq = +\lambda\,dz \qquad\qquad \text{and} \qquad\qquad -dq = -\lambda\,dz \qquad (2c)$$

Here λ is the magnitude of the electric scalar line charge density.

The two magnetic elemental dipoles and the one electric elemental dipole have *di*polar moments of

$$d\boldsymbol{m}_a = d\boldsymbol{m}_{a_+} + d\boldsymbol{m}_{a_-} = \hat{\boldsymbol{\rho}}_3\rho \times \hat{\mathbf{x}}dQ_a + \hat{\boldsymbol{\rho}}_2\rho \times (-\hat{\mathbf{x}}dQ_a) \equiv \mathbf{s}_a \times \hat{\mathbf{x}}dQ_a = \hat{\mathbf{z}}dm_a \qquad (3a)$$

$$dm_b = dm_{b_+} + dm_{b_-} = \hat{\rho}_1 \rho \times \hat{y} dQ_b + \hat{\rho}_2 \rho \times (-\hat{y} dQ_b) \equiv s_b \times \hat{y} dQ_b = \hat{z} dm_b \qquad (3b)$$

$$d\mathbf{p} = d\mathbf{p}_+ + d\mathbf{p}_- = \hat{z} z dq + (-\hat{z} z)(-dq) = s_e dq \equiv \hat{z} dp \qquad (3c)$$

where their intra dipolar displacements or dipolar orientation vectors in Cartesian directions are

$$\mathbf{s}_a = -\hat{y} s_a = -\hat{y} 2\rho \sin\phi \qquad\qquad \mathbf{s}_b = \hat{x} s_b = \hat{x} 2\rho \cos\phi \qquad\qquad \mathbf{s}_e = \hat{z} s_e = \hat{z} 2z \qquad (4)$$

The (sub) subscripts $+$ and $-$ in (3a) to (3c), and in subsequent discussions below, signify entities due to opposing individual elemental vector or scalar charges, that is, *mono*polar contributions.

Integrating (3a) and (3b) from $\phi = 0$ to $\phi = \pi$ and (3c) from $z = 0$ to $z = z$ yield the moments:

$$\mathbf{m}_a = \hat{z} \mu_0 I \pi \rho^2 \equiv \hat{z} m_a \qquad\qquad \mathbf{m}_b = \hat{z} \mu_0 I \pi \rho^2 \equiv \hat{z} m_b \qquad\qquad \mathbf{p} = \hat{z} \lambda z^2 \equiv \hat{z} p \qquad (5)$$

Thus the overall magnetic dipolar moment $\mathbf{m} = \mathbf{m}_a + \mathbf{m}_b = \hat{z} 2\mu_0 I \pi \rho^2$ is twice the traditional value for a circular current [2–8], which would be $\hat{z} \mu_0 I \pi \rho^2$ in the *Kennelly* convention. Note the similarity in the three distinct moments in (5), each involves an apparent area vector, existent or nonexistent.

In external magnetic $\mathbf{H}$ and electric $\mathbf{E}$ fields the Cartesian elemental magnetic and electric dipoles are characterized by paired magnetic forces $\pm d\mathbf{F}_{m_a}$, $\pm d\mathbf{F}_{m_b}$ and electric forces $\pm d\mathbf{F}_e$. As coupled moments of the forces acting on the elemental dipoles, the magnetic and electric torques are:

$$d\boldsymbol{\tau}_a = d\boldsymbol{\tau}_{a_+} + d\boldsymbol{\tau}_{a_-} \equiv \mathbf{s}_a \times d\mathbf{F}_{m_a} = -\hat{y} s_a \times (\hat{x} dQ_a \times \mathbf{H}) = -\hat{y} s_a \times (\hat{z} dQ_a H_y - \hat{y} dQ_a H_z) \qquad (6a)$$

$$d\boldsymbol{\tau}_b = d\boldsymbol{\tau}_{b_+} + d\boldsymbol{\tau}_{b_-} \equiv \mathbf{s}_b \times d\mathbf{F}_{m_b} = +\hat{x} s_b \times (\hat{y} dQ_b \times \mathbf{H}) = +\hat{x} s_b \times (\hat{x} dQ_b H_z - \hat{z} dQ_b H_x) \qquad (6b)$$

$$d\boldsymbol{\tau}_e = d\boldsymbol{\tau}_{e_+} + d\boldsymbol{\tau}_{e_-} = \mathbf{s}_e \times d\mathbf{F}_e = \hat{z} s_e \times (dq\mathbf{E}) = \hat{z} s_e \times (\hat{z} dq E_z + \hat{y} dq E_y + \hat{x} dq E_x) \qquad (6c)$$

Note the triple vector products in (6a) and (6b), and the duo vector product in (6c). The Cartesian components of the torques become

$$d\boldsymbol{\tau}_{m_x} = d\boldsymbol{\tau}_a = \hat{z} s_a dQ_a \times \hat{y} H_y \equiv \{\mathbf{s}_a \times \hat{x} dQ_a\} \times \hat{y} H_y = d\mathbf{m}_a \times \hat{y} H_y \neq d\mathbf{m}_a \times \mathbf{H}$$

$$d\boldsymbol{\tau}_{m_y} = d\boldsymbol{\tau}_b = \hat{z} s_b dQ_b \times \hat{x} H_x \equiv \{\mathbf{s}_b \times \hat{y} dQ_b\} \times \hat{x} H_x = d\mathbf{m}_b \times \hat{x} H_x \neq d\mathbf{m}_b \times \mathbf{H}$$

$$d\boldsymbol{\tau}_{e_x} \qquad = \hat{z} s_e dq \times \hat{y} E_y \quad \equiv \quad \{\mathbf{s}_e dq\} \times \hat{y} E_y \;=\; d\mathbf{p} \times \hat{y} E_y \;\neq\; d\mathbf{p} \times \mathbf{E}$$

$$d\boldsymbol{\tau}_{e_y} \qquad = \hat{z} s_e dq \times \hat{x} E_x \quad \equiv \quad \{\mathbf{s}_e dq\} \times \hat{x} E_x \;=\; d\mathbf{p} \times \hat{x} E_x \;\neq\; d\mathbf{p} \times \mathbf{E}$$

$$(7)$$

Note the matched inequalities. Using (5), the overall magnetic and electric torques become

$$\boldsymbol{\tau}_m = \boldsymbol{\tau}_a + \boldsymbol{\tau}_b = \mathbf{m}_a \times \hat{y} H_y + \mathbf{m}_b \times \hat{x} H_x \equiv \hat{z} m_a \times (\hat{\rho} H_\rho + \hat{z} H_z) \equiv \mathbf{m}_a \times \mathbf{H} \neq \mathbf{m} \times \mathbf{H} \qquad (8)$$

$$\boldsymbol{\tau}_e = \boldsymbol{\tau}_{e_x} + \boldsymbol{\tau}_{e_y} = \mathbf{p} \times \hat{y} E_y + \mathbf{p} \times \hat{x} E_x \equiv \hat{z} p \times (\hat{\rho} E_\rho + \hat{z} E_z) \equiv \mathbf{p} \times \mathbf{E} \neq 2\mathbf{p} \times \mathbf{E} \qquad (9)$$

Thus the traditional choice [2–8] of $\mathbf{m}_a$ or $\mathbf{m}_b$ as the total magnetic dipolar moment is unjustified. The inequalities in (8) and (9) nullify the traditional analogy between magnetic and electric torques.

3. Dipolar magnetic vector and electric scalar potentials and associated fields

When the magnetic vector and electric scalar charge distributions in figure 1 are the sources of magnetic vector and electric scalar potentials, as well as the associated magnetic and electric fields at a field point P, the charges and their positions are signified by primed symbols. Thus, in figure 2 the field point P in a $z\rho$-plane is at position $\mathbf{r} = \hat{r} r$ from the origin O and its displacements from the source elemental Cartesian magnetic vector charges at points P'_j, $j = 1, 2, 3$ on the circle are

$$\mathbf{R}_j = \hat{\mathbf{R}}_j R_j = \hat{\mathbf{R}}_j f_j^{\frac{1}{2}} r = \hat{\mathbf{R}}_j (1 + \eta_j)^{\frac{1}{2}} r = \mathbf{r} - \boldsymbol{\rho}'_j \equiv \hat{r} r - \hat{\boldsymbol{\rho}}'_j \rho' = \hat{z} z + \hat{\rho} \rho - \hat{\boldsymbol{\rho}}'_j \rho', \quad j = 1, 2, 3 \qquad (10)$$

where $\rho = r \sin\theta$ and the geometrical factor f_j (or η_j) is a function of r, θ, ϕ, ρ', ϕ'. Similarly in figure 3, displacements of the field point P from the electric scalar charges $+ dq'$ and $- dq'$ at P'_+ and P'_- on the z-axis are
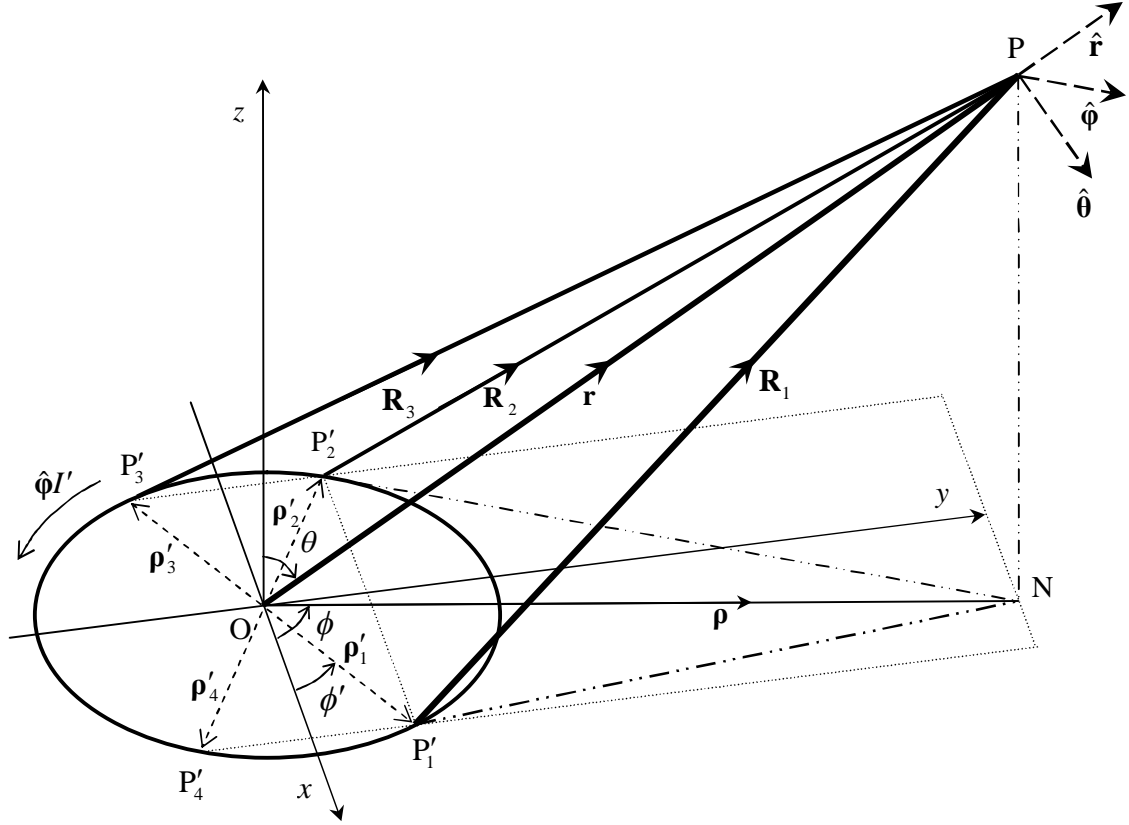
Figure 2: A field point P in a $z\rho$-plane and magnetic source points P'_1, P'_2, P'_3, P'_4 on an xy-circle.

$$\mathbf{R}_{\pm} = \hat{\mathbf{R}}_{\pm} R_{\pm} = \hat{\mathbf{R}}_{\pm} f_{\pm}^{\frac{1}{2}} r = \hat{\mathbf{R}}_{\pm} (1+\eta_{\pm})^{\frac{1}{2}} r = \mathbf{r} - \mathbf{z}'_{\pm} = \hat{\mathbf{r}}\left(r \mp \tfrac{1}{2} s'_e \cos\theta\right) \pm \hat{\boldsymbol{\theta}}\tfrac{1}{2} s'_e \sin\theta \tag{11}$$

where $s'_e = 2z'$, with the geometrical factor $f_{\pm}$ (or $\eta_{\pm}$) being a function of r, θ, z' only.

Below, each ensuing first order approximation for $\rho' << r$ or $z' << r$ is after binomial expansion of $f_j^{-\frac{n}{2}}$ or $f_{\pm}^{-\frac{n}{2}}$, where $n=1$ for potentials and $n=3$ for fields, and application of equations (3a) to (3c). Then the two Cartesian elemental magnetic dipoles on the circle in figure 2 and the electric dipole on the z-axis in figure 3 will generate at P the magnetic vector and electric scalar potentials

$$d\mathbf{A}_a = d\mathbf{A}_{a_+} + d\mathbf{A}_{a_-} = \left(f_3^{-\frac{1}{2}} - f_2^{-\frac{1}{2}}\right)\frac{\hat{\mathbf{x}}dQ'_a}{4\pi\mu_0 r} \approx -\hat{\mathbf{x}}\sin\phi\sin\theta\frac{s'_a dQ'_a}{4\pi\mu_0 r^2} \equiv -\hat{\mathbf{x}}\sin\phi\sin\theta\frac{dm'_a}{4\pi\mu_0 r^2} \tag{12a}$$

$$d\mathbf{A}_b = d\mathbf{A}_{b_+} + d\mathbf{A}_{b_-} = \left(f_1^{-\frac{1}{2}} - f_2^{-\frac{1}{2}}\right)\frac{\hat{\mathbf{y}}dQ'_b}{4\pi\mu_0 r} \approx +\hat{\mathbf{y}}\cos\phi\sin\theta\frac{s'_b dQ'_b}{4\pi\mu_0 r^2} \equiv +\hat{\mathbf{y}}\cos\phi\sin\theta\frac{dm'_b}{4\pi\mu_0 r^2} \tag{12b}$$

$$dV = dV_+ + dV_- = \left(f_+^{-\frac{1}{2}} - f_-^{-\frac{1}{2}}\right)\frac{dq}{4\pi\varepsilon_0 r} \approx \cos\theta\frac{s'_e dq}{4\pi\varepsilon_0 r^2} \equiv \cos\theta\frac{dp'}{4\pi\varepsilon_0 r^2} \tag{12c}$$

Integrating and changing from Cartesian to spherical unit vectors shows that each of $\mathbf{A}_a$ and $\mathbf{A}_b$ varies with ϕ, but V does not:

$$\mathbf{A}_a = \left[-\left(\hat{\mathbf{r}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta\right)\sin\phi\cos\phi + \hat{\boldsymbol{\phi}}\sin^2\phi\right]\frac{m'_a \sin\theta}{4\pi\mu_0 r^2} \tag{13a}$$

$$\mathbf{A}_b = \left[+ \left(\hat{\mathbf{r}} \sin\theta + \hat{\boldsymbol{\theta}} \cos\theta \right) \sin\phi \cos\phi + \hat{\boldsymbol{\phi}} \cos^2\phi \right] \frac{m_b' \sin\theta}{4\pi\mu_0 r^2} \tag{13b}$$

$$V = \frac{p' \cos\theta}{4\pi\varepsilon_0 r^2} \tag{13c}$$

Due to the equations in (5), the total magnetic vector potential become

$$\mathbf{A} = \mathbf{A}_a + \mathbf{A}_b \equiv \left(-\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi \right) \frac{m_a'}{4\pi\mu_0 r^2} = \hat{\boldsymbol{\phi}} \sin\theta \frac{m_a'}{4\pi\mu_0 r^2} \equiv \hat{\boldsymbol{\phi}} \sin\theta \frac{m_b'}{4\pi\mu_0 r^2} \tag{14}$$
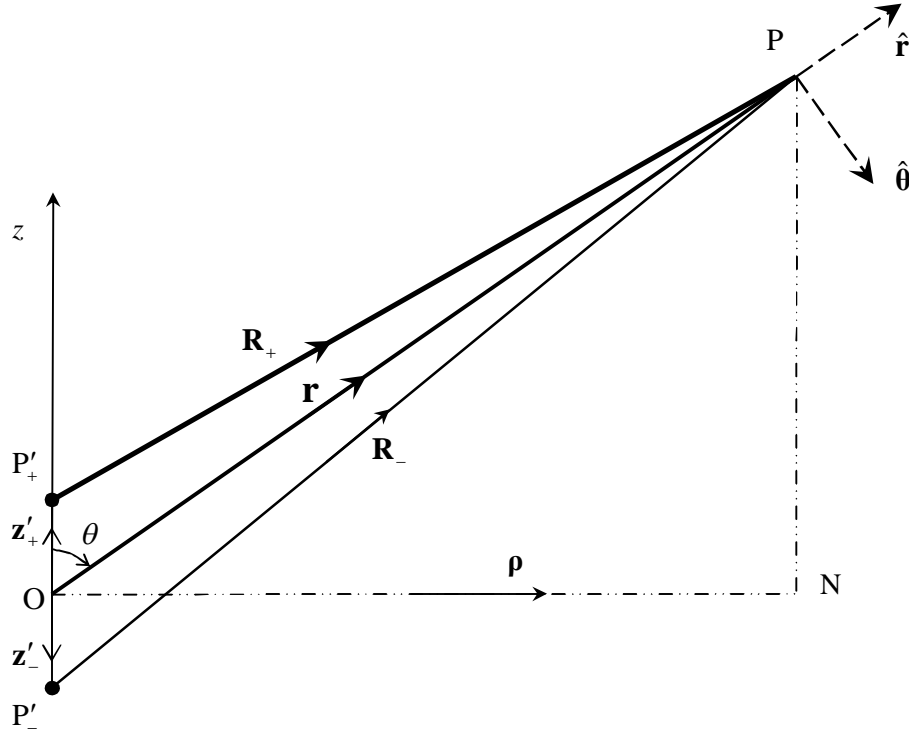

Figure 3: A field point P in a $z\rho$-plane or $r\theta$-plane and electric source points P_+', P_-' on a z axis.

Similarly, the fields at point P due to the two distinct Cartesian elemental magnetic dipoles in figure 2 and the electric dipole in figure 3 are

$$d\mathbf{H}_a = d\mathbf{H}_{a_+} + d\mathbf{H}_{a_-} = \frac{\hat{\mathbf{x}} dQ_a'}{4\pi\mu_0 r^3} \times \left(\mathbf{R}_3 f_3^{-\frac{3}{2}} - \mathbf{R}_2 f_2^{-\frac{3}{2}} \right)$$
$$\approx \left[\hat{\mathbf{y}} 3 \sin\phi \cos\theta \sin\theta + \hat{\mathbf{z}} \left(1 - 3\sin^2\phi \sin^2\theta \right) \right] \frac{s_a' dQ_a'}{4\pi\mu_0 r^3} \tag{15a}$$

$$d\mathbf{H}_b = d\mathbf{H}_{b_+} + d\mathbf{H}_{b_-} = \frac{\hat{\mathbf{y}} dQ_b'}{4\pi\mu_0 r^3} \times \left(\mathbf{R}_1 f_1^{-\frac{3}{2}} - \mathbf{R}_2 f_2^{-\frac{3}{2}} \right)$$
$$\approx \left[\hat{\mathbf{x}} 3 \cos\phi \cos\theta \sin\theta + \hat{\mathbf{z}} \left(1 - 3\cos^2\phi \sin^2\theta \right) \right] \frac{s_b' dQ_b'}{4\pi\mu_0 r^3} \tag{15b}$$

$$d\mathbf{E} = d\mathbf{E}_+ + d\mathbf{E}_- = \frac{dq'}{4\pi\varepsilon_0 r^3} \left(\mathbf{R}_{e_+} f_+^{-\frac{3}{2}} - \mathbf{R}_{e_-} f_-^{-\frac{3}{2}} \right) \approx \left(\hat{\mathbf{r}} 2\cos\theta + \hat{\boldsymbol{\theta}} \sin\theta \right) \frac{s_e' dq'}{4\pi\varepsilon_0 r^3} \tag{15c}$$

Integrating (15a) to (15c), and transforming unit vectors from Cartesian to spherical systems shows that both $\mathbf{H}_a$ and $\mathbf{H}_b$ vary with ϕ, but $\mathbf{E}$ does not:

$$\mathbf{H}_a = \left[\hat{\mathbf{r}}\cos\theta + \hat{\boldsymbol{\theta}}\left(3\sin^2\phi - 1\right)\sin\theta + \hat{\boldsymbol{\varphi}}3\cos\phi\sin\phi\cos\theta\sin\theta\right]\frac{m_a'}{4\pi\mu_0 r^3} \tag{16a}$$

$$\mathbf{H}_b = \left[\hat{\mathbf{r}}\cos\theta + \hat{\boldsymbol{\theta}}\left(3\cos^2\phi - 1\right)\sin\theta - \hat{\boldsymbol{\varphi}}3\cos\phi\sin\phi\cos\theta\sin\theta\right]\frac{m_b'}{4\pi\mu_0 r^3} \tag{16b}$$

$$\mathbf{E} = (\hat{\mathbf{r}}2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta)\frac{p'}{4\pi\varepsilon_0 r^3} \tag{16c}$$

Then as $m_a' = m_b'$ (equations 5), the overall magnetic field acquires cylindrical symmetry when expressed exclusively in terms of either m_a' or m_b':

$$\mathbf{H} = \mathbf{H}_a + \mathbf{H}_b = \frac{m_a'}{4\pi\mu_0 r^3}\left(2\hat{\mathbf{r}}\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta\right) \equiv \frac{m_b'}{4\pi\mu_0 r^3}\left(2\hat{\mathbf{r}}\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta\right) \tag{17}$$

Clearly the similarity between (15c) and (17) cannot justify the tradition [2−8] of taking $\boldsymbol{m}_a'$ or $\boldsymbol{m}_b'$ as the circular current's *only* magnetic moment. Again the traditional analogy fails.

4. Conclusions

It has been shown that traditional analogies between the structures and torques of electric and magnetic dipoles are deceptively erroneous. A circular current is resolvable into merged distributions of distinct Cartesian *x* and *y* component line current elements, each distribution being a complete magnetic dipole that selectively creates *like* Cartesian components of the magnetic torque and *azimuthal* magnetic vector potential, plus only the magnetic field's *other* Cartesian components. All these are expressible in terms of a distribution's *own* magnetic dipolar moment, which is traditionally attributed to the *whole* circular current. In contrast a simple electric dipole aligned on the *z*-axis, creates its *x* and *y* electric torque components, its full cylindrically symmetric electric field and the electric scalar potential, all of which are expressible in terms of the sole electric dipolar moment.

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